AAS 17-122

ANALYTICAL ATTITUDE DETERMINITION FROM A SPECIFIC RATE PROFILE

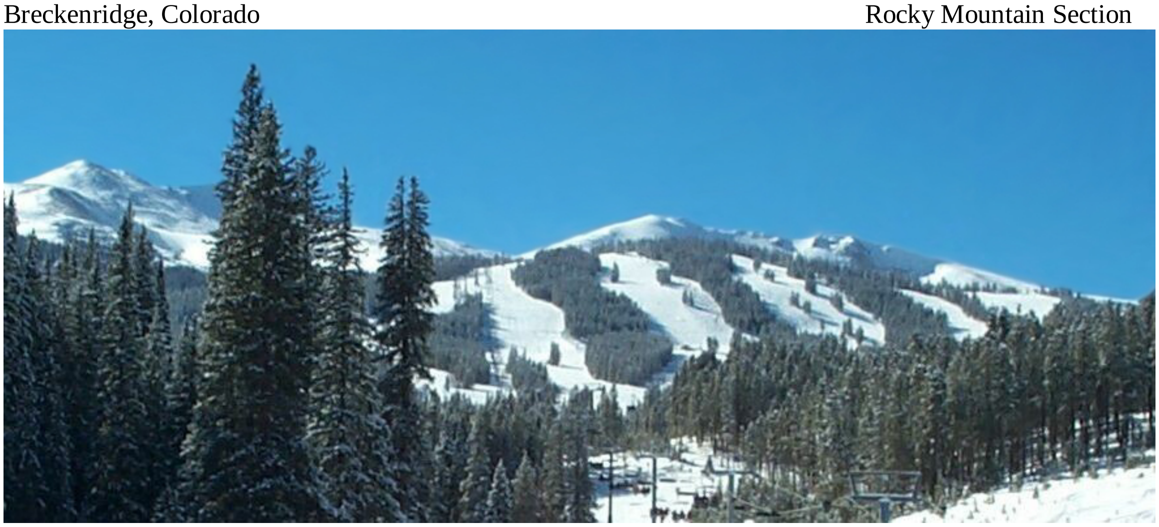
Prof. Dr. Peter Zentgraf[1]

[1]University of Applied Sciences Rosenheim

40th ANNUAL AAS GUIDANCE AND CONTROL CONFERENCE

February 2 – February 8, 2017
Breckenridge, Colorado

Sponsored by
Rocky Mountain Section



AAS Publications Office, P.O. Box 28130 - San Diego, California 92198

AAS 17-122

ANALYTICAL ATTITUDE DETERMINATION FROM A SPECIFIC RATE PROFILE

Prof. Dr. Peter Zentgraf[1]

The problem dealt with in this paper is the analytical determination of the attitude of a rigid body which undergoes a given rate profile. Usually the attitude is determined by solving the kinematic differential equation numerically. For long lasting movements the outcome can cost computation time plus encountered losses in accuracy, which can be disturbing especially in optimization problems. This contribution solves the kinematic differential equation for certain rate profiles exactly and compares the results in accuracy and computation time with the standard procedures.

INTRODUCTION

During missions of Earth observation satellites one important task is to scan predefined spots on the surface. In order to do so the satellite needs to slew from one particular initial attitude and a given rate to the new target defined by a new attitude and a corresponding rate, see for instance Figure 1. A method to predict the achieved attitude from a given rate profile analytically in a computationally fast way is shown in this paper.

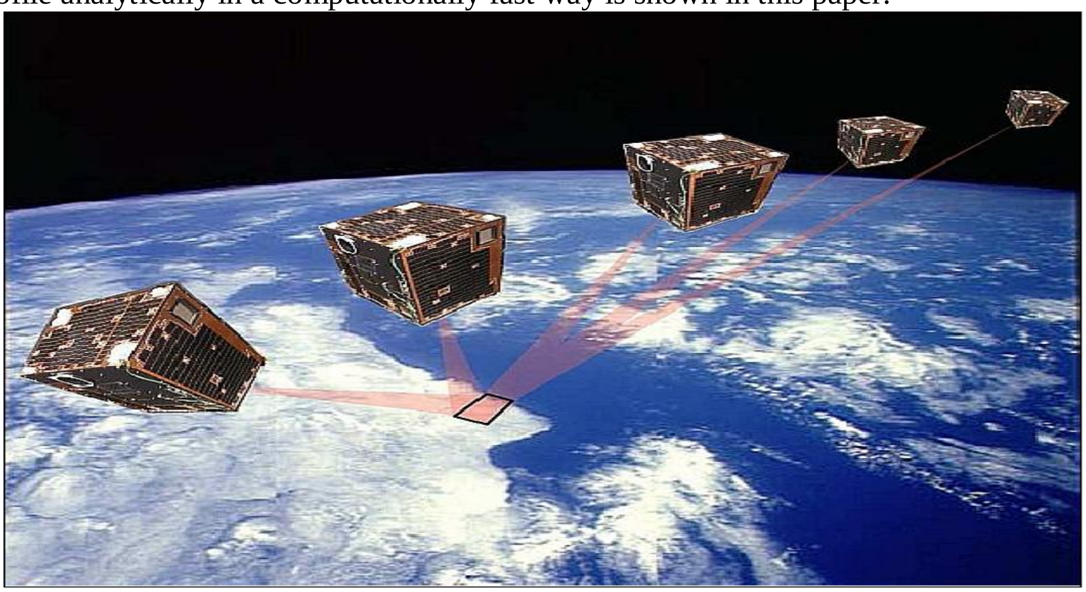


Figure 1: Illustration of a multi-angle observation sequence of PROBA-1 (image from ESA EO-directory web-portal)

[1]University of Applied Sciences – Hochschulstrasse 1, 83024 Rosenheim, Germany,
Email: peter.zentgraf@fh-rosenheim.de

PROBLEM STATEMENT

In this section the following problem is considered: For given initial constraints ω_0, T_0 at time instant t=0s and the final rate ω_1 that is reached at given time instant t=t_1 find an analytical solution to determine the corresponding final attitude $\boldsymbol{T_1}$ t=t_1 as shown in **Figure 2.**

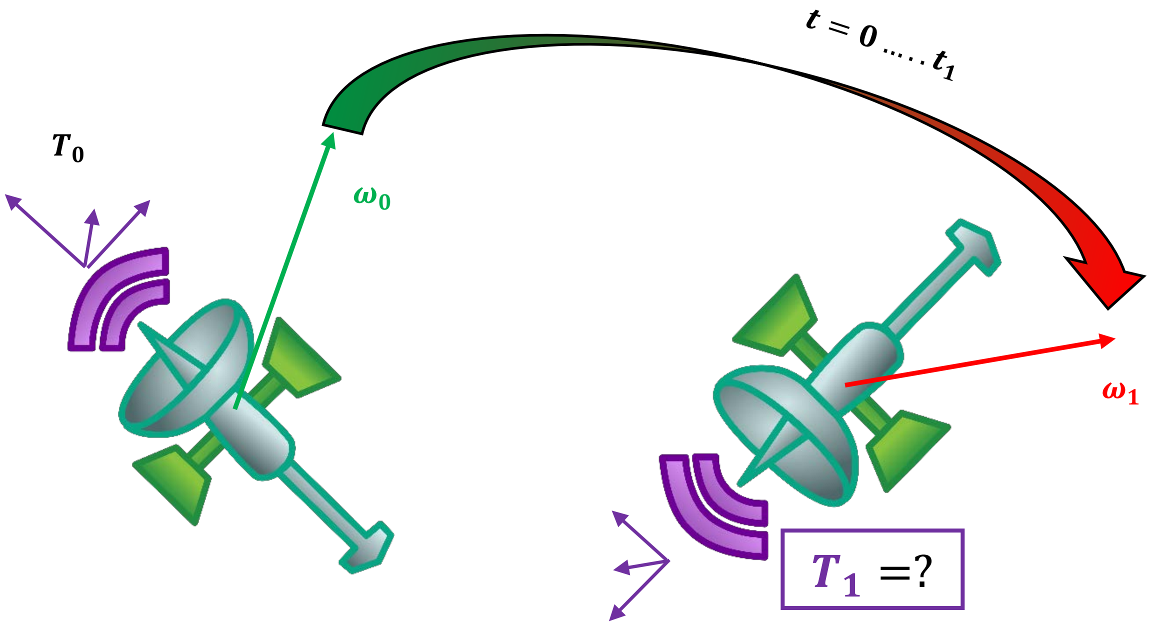


Figure 2: Determine final attitude $\boldsymbol{T_1}$ from a rate change of $\boldsymbol{\omega_0}$ to $\boldsymbol{\omega_1}$

Let us first consider the change of a vector r(t) – which can be imagined fixed to a body - during a small time period $\Delta \boldsymbol{t}$ within the inertial frame $\boldsymbol{i}$ due to the rotation around vector $\boldsymbol{\omega}(\boldsymbol{t})$ as shown in Figure 3, which leads to the differential equation eq.(1),

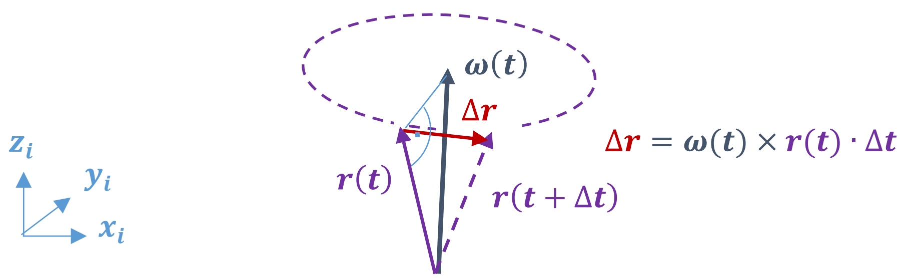


Figure 3: Change of a vector due to rotation

$$\boldsymbol{r}' = \widetilde{\boldsymbol{\omega}} \cdot \boldsymbol{r} \qquad (1)$$

Note that $\boldsymbol{\omega}(\boldsymbol{t})$ is the rotation of the body frame w.r.t. inertial frame and is also expressed in the inertial frame. Now consider not only one vector $r(t)$ but three orthonormal vectors

x_b, y_b, z_b spanning the body fixed frame vectors collected in the transformation from the body-frame into the inertial frame

$$T_b^i(t) = \begin{bmatrix} | & | & | \\ x_b(t) & y_b(t) & z_b(t) \\ | & | & | \end{bmatrix} \tag{2}$$

$$T_b^i(t)' = \widetilde{\omega} \cdot T_b^i(t) \tag{3}$$

Eq. (3) is the so called kinematic differential equation. Note that it differs from the corresponding equation in [2], because the derivative refers to the body frame and not to the inertial frame as it refers in here, and it is expressed in [2] as $T_i^b(t)$ rather than as $T_b^i(t)$ as it is expressed in here.

The problem is now to solve eq. (3) analytically and to yield an expression $T_i^b(t)$, meaning to find an explicit formula which predicts the attitude w.r.t. time t for a given rate profile $\boldsymbol{\omega}(\boldsymbol{t})$. Unfortunately this cannot be solved analytically for any rate profile $\boldsymbol{\omega}(\boldsymbol{t})$, but for some special rate profiles this can be done.

A special rate profile

The proposed method is inspired by [1], in which the closed form solution for an angular rate vector of constant magnitude is presented that is slewing at a constant angular rate. The method presented here works for any magnitude of the rate vector.
Consider the profile of the rate vector $\omega(t)$ starting from $t = 0$ with $\boldsymbol{\omega_0}$ as it rotates itself about the time fixed rate vector rotation axis defined by the unit vector Ω_u. The instantaneous component of $\boldsymbol{\omega}(t)$ with respect to $\boldsymbol{\Omega_u}$ is vector $\boldsymbol{\Omega}$ as shown in Figure 4.

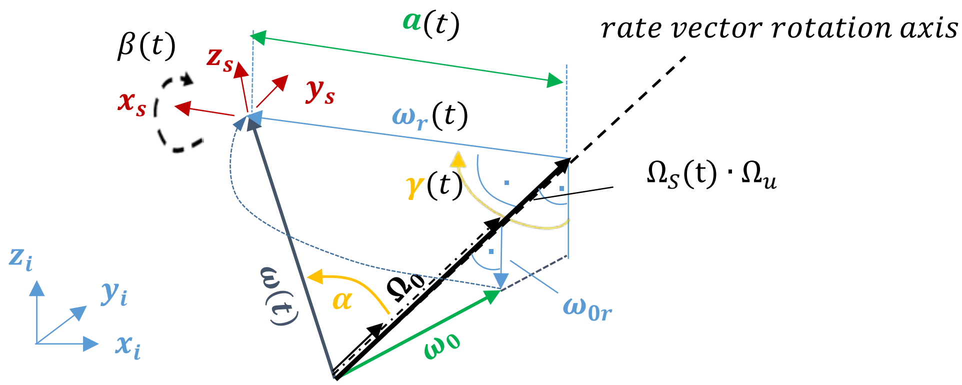


Figure 4: Geometry of the considered rate profile

So, the vector $\boldsymbol{\omega}(t)$ is decomposed from Cartesian coordinates $\omega_x(t), \omega_y(t), \omega_z(t)$ into cylindrical coordinates, “height” $\Omega_S(t)$, “radius” $\boldsymbol{a}(t)$ and angle $\boldsymbol{\gamma}(t)$ as shown in Figure 5.
Angle $\boldsymbol{\gamma}$ is defined as the integrated rate component of $\omega(t)$ w.r.t Ω_u, i.e. the scalar product of Ω_u and $\boldsymbol{\omega}(\boldsymbol{t}),$ $\Omega_u{}^T \cdot \boldsymbol{\omega}(\boldsymbol{t}),$ in which "T" denotes the “transpose” of a vector,

$$\gamma(t) = \int_0^t \boldsymbol{\Omega_u}^T \cdot \boldsymbol{\omega}(\boldsymbol{t})\, dt = \int_0^t \Omega_s(\mathrm{t}) \cdot dt \tag{4}$$

In eq.(4), $\Omega_s(\mathrm{t})$ describes the scalar rate component w.r.t the direction of $\boldsymbol{\Omega_u}$,

$$\Omega_s(\mathrm{t}) := \boldsymbol{\Omega_u}^T \cdot \boldsymbol{\omega}(t) \tag{5}$$

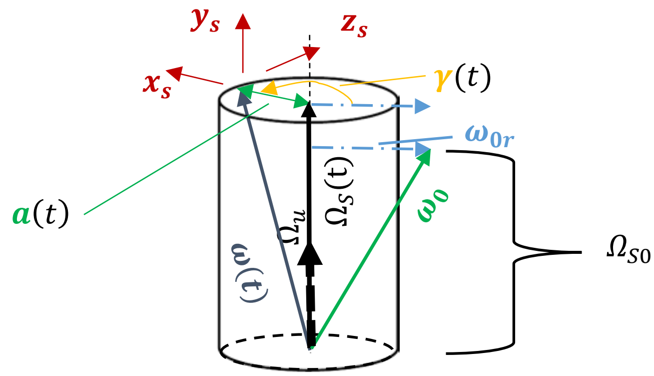


Figure 5: Transformation from Cartesian coordinates into cylindrical coordinates

and $\Omega_s(\mathrm{t})$ may have positive or negative sign. The instantaneous rate vector $\boldsymbol{\Omega}(\boldsymbol{t})$ can also be expressed using time invariant unit vector Ω_u and the time dependent component $\Omega_s(\mathrm{t})$ by

$$\boldsymbol{\Omega}(\boldsymbol{t}) = \boldsymbol{\Omega_u} \cdot \Omega_s(t) \tag{6}$$

From definition in eq.(4), angle $\gamma(t=0) = 0$ at the initial rate $\boldsymbol{\omega}(\boldsymbol{t} = \mathbf{0}) = \boldsymbol{\omega_0}$. Eq. (4) shows also that there is no freedom in the choice of $\gamma(t)$: Once $\Omega_s(\mathrm{t})$ is selected, the path of $\gamma(t)$ is uniquely defined. This is an important restriction: Not all possible rate profiles $\omega(t)$ are admitted, because if so then there would be no restriction on $\gamma(t)$. The special rate profile allows freedom in the choice of "height"-function $\Omega_S(\mathrm{t})$ and "radius" $\boldsymbol{a}(t)$, but the third dimension of the rate profile, angle $\gamma(t)$, is fixed by the choice of $\Omega_s(\mathrm{t})$ as shown in Figure 6 with initial and final height"-function values, Ω_{S0} and Ω_{S1}, respectively.

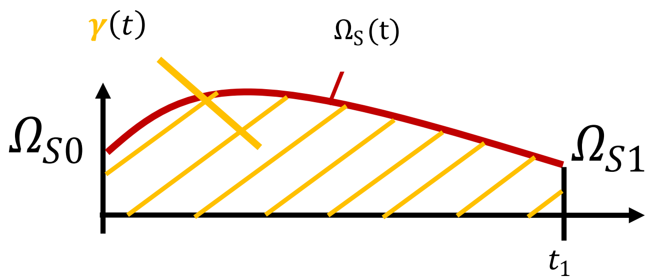


Figure 6: Angle $\gamma(t)$-restriction from special cylinder height function $\Omega_s(\mathrm{t})$

This is the price to be paid to solve the kinematic differential equation; how this is done is shown next.

A new "support frame" s is introduced into which the kinematic differential equation will be transformed to. Its basic vectors are defined as follows:

1. basic vector $y_s = \Omega_u$

2. basic vector x_s: Its direction is defined by vector $\boldsymbol{\omega_{0r}}$, which is rotated around Ω_u with angle $\gamma(t)$, see Figure 4 and Figure 5.
3. basic vector $z_s = x_s \times y_s$

Vector x_s is perpendicular to $\boldsymbol{\Omega}$. The corresponding cylindrical component, $a(t)$, may have positive or negative sign, so it is more general than an “always-positive-‘radial’-component”. Now the “special” rate vectors $\omega(t)$ can be expressed:

$$\boldsymbol{\omega}(t) = a(t) \cdot \boldsymbol{x_s} + \Omega_s(t) \cdot \boldsymbol{y_s} \tag{7}$$

The considered rate $\omega(t)$ is not allowed to ever have a component into $\boldsymbol{z_s}$- direction. Component $\boldsymbol{a}(t)$ refers to the direction of $\boldsymbol{x_s}$ and may also have a negative sign, i.e.

$$a(t) = \boldsymbol{x_s}^T \cdot \boldsymbol{\omega}(t) \tag{8}$$

Alternatively, the magnitude of $a(t)$, $|a(t)|$, can simply be derived from Figure 4 as

$$|a(t)| = |sin\, \alpha \cdot \omega| = \left|\left(E - \boldsymbol{\Omega_u} \cdot \boldsymbol{\Omega_u}^T\right) \cdot \boldsymbol{\omega}(t)\right| \tag{9}$$

where $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the unit matrix.

Solving the kinematic differential equation in the support frame

Recall the objective which is to solve eq. (1), i.e. to determine vector $\boldsymbol{r}$ in an explicit and analytical formula.
Expressing $\boldsymbol{r}$ in the support frame ” s” as vector v and taking the derivative (symbol “ ’ ”) with respect to the inertial frame yields

$$r = T_s^i \cdot v \tag{10}$$

$$r' = T_s^{i'} \cdot v + T_s^i \cdot v' \tag{11}$$

In there, the transformation from the support-frame s into the inertial frame i is given by the transformation matrix

$$T_s^i = \begin{bmatrix} | & | & | \\ x_s & y_s & z_s \\ | & | & | \end{bmatrix}. \tag{12}$$

Similar to Figure 3 the change of vector x_s , x_s', where x_s rotates around vector $\Omega(\mathrm{t})$ yields

$$x_s' = \tilde{\Omega} \cdot x_s \tag{13}$$

and similar for the complete base x_s, y_s, z_s as matrix equation

$$T_s^{i'} = \tilde{\Omega} \cdot T_s^i \tag{14}$$

Inserting now eqs. (14) into (11), and then (11) and (10) into (1) gives

$$\tilde{\Omega}\cdot T_s^i\cdot v+T_s^i\cdot v' = \tilde{\omega}\cdot T_s^i\cdot v$$
$$T_s^i\cdot v' = \left(\tilde{\omega}-\tilde{\Omega}\right)\cdot T_s^i\cdot v$$
$$v' = T_s^{i\,T}\cdot\left(\tilde{\omega}-\tilde{\Omega}\right)\cdot T_s^i\cdot v$$
$$v' = T_s^{i\,T}\cdot\left(\tilde{\omega}-\tilde{\Omega}\right)\cdot\begin{bmatrix} | & | & | \\ x_s & y_s & z_s \\ | & | & | \end{bmatrix}\cdot v$$
$$v' = T_s^{i\,T}\cdot\begin{bmatrix} | & | & | \\ 0 & |\omega-\Omega|\cdot z_s & -|\omega-\Omega|\cdot y_s \\ | & | & | \end{bmatrix}\cdot v$$
$$v' = \begin{bmatrix} - & {x_s}^T & - \\ - & {y_s}^T & - \\ - & {z_s}^T & - \end{bmatrix}\cdot\begin{bmatrix} | & | & | \\ 0 & z_s & -y_s \\ | & | & | \end{bmatrix}\cdot|\omega-\Omega|\cdot v$$
$$v'(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\cdot \boldsymbol{a}(t)\cdot v(t). \tag{15}$$

Eq. (15) can now be solved analytically, since the involved matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ is not depending on time. Rearranging yields

$$v(t) = e^{\int_0^t\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\cdot\boldsymbol{a}(t)\right)dt}\cdot\underbrace{v(t=0)}_{=:v_0} = e^{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\cdot\underbrace{\int_0^t(\boldsymbol{a}(t))\,dt}_{=:\beta(t)}}\cdot v_0. \tag{16}$$

The exponential matrix in eq. (16) can be solved analytically to

$$v(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta(t) & -\sin\beta(t) \\ 0 & \sin\beta(t) & \cos\beta(t) \end{bmatrix}}_{=:T_{x,-\beta}}\cdot v_0; \tag{17}$$

the definition of the rotation, $T_{x,-\beta}$, contains a minus sign in order to be compatible with the Euler rotation matrices. Thus, the change of vector $v(t)$ is a pure rotation (rotation matrix $T_{x,-\beta}$) with angle β about the x-axis of the support frame, see also Figure 4, where β is defined as

$$\beta(t) = \int_0^t \boldsymbol{a}(t)\cdot dt. \tag{18}$$

Inserting the solution $v(t)$ in eq. (10) yields

$$r(t) = T_s^i(t)\cdot T_{x,-\beta}\cdot T_s^{i\,T}(t=0)\cdot r_0. \tag{19}$$

The supporting frame is rotating with angle γ around the local y-axis and can be expressed as

$$T_s^i(t) = T_s^i(0) \cdot \underbrace{\begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}}_{T_{y,-\gamma}}. \tag{20}$$

Inserting eq. (20) in (19) gives finally

$$\boldsymbol{r}(t) = T_s^i(0) \cdot T_{y,-\gamma(t)} \cdot T_{x,-\beta(t)} \cdot {T_s^i}^T(0) \cdot \boldsymbol{r_0}. \tag{21}$$

This means, the new direction of $r(t)$ can be seen within the supporting frame as a rotation of the initial vector around the local x-axis by angle β and then as a rotation around the y-axis by angle γ.

Extending according to (2) gives then finally

$$T_b^i(t) = T_s^i(0) \cdot T_{y,-\gamma(t)} \cdot T_{x,-\beta(t)} \cdot {T_s^i}^T(0) \cdot {T_b^i}_0. \tag{22}$$

The rates can be expressed similarly: Inserting eq. (20) into eq.(7) to transform the rate from the support frame back into the inertial frame yields

$$\boldsymbol{\omega}(t) = \boldsymbol{\Omega_u} \cdot \Omega_s(t) + T_s^i(0) \cdot T_{y,-\gamma(t)} \cdot \begin{bmatrix} a(t) \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

For *t=0* eq. (23) simplifies as seen in Figure 4 to

$$\boldsymbol{\omega}(0) = \Omega_u \cdot \Omega_s(0) + T_s^i(0) \cdot \begin{bmatrix} a(0) \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{\Omega_0} + T_s^i(0) \cdot \begin{bmatrix} |\omega_{0r}| \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{\Omega_0} + \omega_{0r} = \omega_0. \tag{24}$$

thus yielding the initial rate $\boldsymbol{\omega_0}$.

Preselecting initial rate $\boldsymbol{\omega_0}$, final rate $\boldsymbol{\omega_1}$ and slew time $\boldsymbol{t_1}$

Selecting any rotation vector $\boldsymbol{\Omega_u}$ determines initial and final cylindrical coordinates, see Figure 4 and Figure 7, because then

- the rotation angle $\gamma(t_1)$ is fixed,
- the radial rate component $a_0 = a(0)$ (and thus namely $\boldsymbol{x_s}(t=0)$ in eq. (12)) and $a_1 = a(t_1)$ are fixed,
- the choice of any $\boldsymbol{\Omega_u}$ fixes by itself the axial component of $\boldsymbol{\omega_1}$, i.e. $\Omega_{s1} = \Omega_s(t_1)$ in eq. (23) (and so $\boldsymbol{y_s}(t=0)$) and also the axial component of $\boldsymbol{\omega_0}$ i.e. $\Omega_{s0} = \Omega_s(0)$ in eq. (23).

In addition, the axial scalar rate component Ω_s and the axial rotation angle component rotation angle $\gamma(t_1)$ are interconnected by the integral restriction eq. (4), i.e. $\gamma(t_1) = \int_0^{t_1} \Omega_s(\mathrm{t}) \cdot dt$. This means, that for a given slew time t_1 the rate component $\Omega_s(\mathrm{t})$ needs to be shaped accordingly such that its integral is equal to angle $\gamma(t_1)$. This also means that only two of three *cylindrical coordinates* are independent. For the radial rate components $a(0)$ and $a(t_1)$ a similar integral-restriction does not exist.

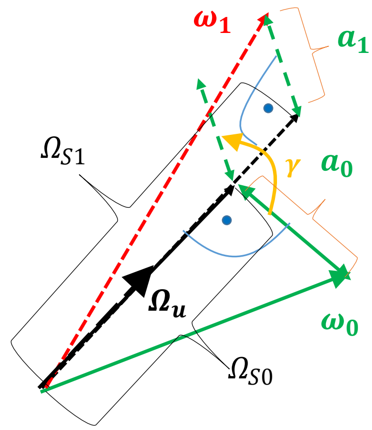


Figure 7:Preselecting initial rate ω_0, final rate ω_1 and slew time t_1 with corresponding scalar cylinder coordinates γ, a_0, a_1, Ω_{s0}, Ω_{s1}

Determination of a unit vector Ω_u for linear change restriction.

To simplify the transformation from Cartesian to cylindrical coordinates it is shown that for many cases a unit rotation rate vector $\boldsymbol{\Omega_u}$ can be found that slews the rate vector $\boldsymbol{\omega}(t)$ from $\boldsymbol{\omega}(0)$=ω_0 to $\boldsymbol{\omega}(t_1) = \boldsymbol{\omega_1}$ about the unit rotation rate vector $\boldsymbol{\Omega_u}$ while the *cylinder coordinates* $\Omega_s(\mathrm{t})$ and $a(t)$ undergo a linear time change as shown in Figure 8.

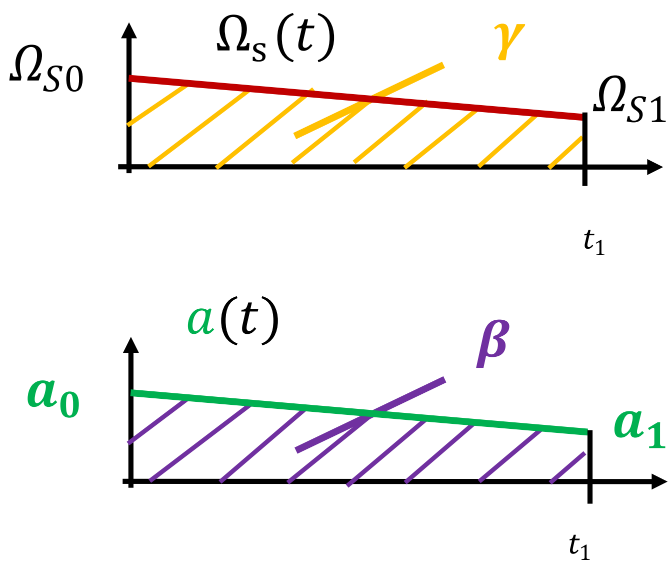


Figure 8: Consideration of linear change in the cylinder coordinates $\Omega_s(\mathrm{t})$ and $a(t)$

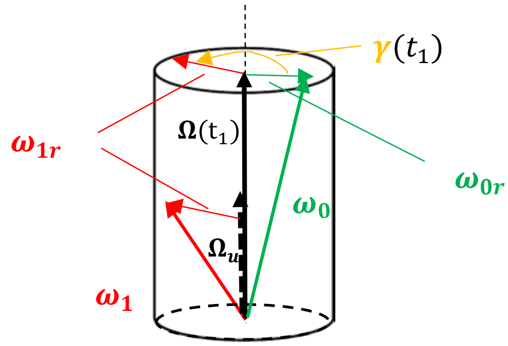


Figure 9: Transformation from Cartesian coordinates into cylindrical coordinates

From Figure 6, the usual cosine law can be written as

$$\boldsymbol{\omega}_{0r}{}^{T} \cdot \boldsymbol{\omega}_{1r} = |\boldsymbol{\omega}_{0r}| \cdot |\boldsymbol{\omega}_{1r}| \cdot \cos \gamma \tag{25}$$

where

$$\boldsymbol{\omega}_{0r} = \left(\boldsymbol{E} - \boldsymbol{\Omega}_u \cdot \boldsymbol{\Omega}_u{}^{T}\right) \cdot \boldsymbol{\omega}_0 \tag{26}$$

$$\boldsymbol{\omega}_{1r} = \left(\boldsymbol{E} - \boldsymbol{\Omega}_u \cdot \boldsymbol{\Omega}_u{}^{T}\right) \cdot \boldsymbol{\omega}_1. \tag{27}$$

Inserting eqs. (27)-(28) into eq. (26) yields

$$\boldsymbol{\omega}_0{}^{T} \cdot \left(\boldsymbol{E} - \boldsymbol{\Omega}_u \cdot \boldsymbol{\Omega}_u{}^{T}\right) \cdot \boldsymbol{\omega}_1 =\cdot \sqrt{\boldsymbol{\omega}_0{}^{T} \cdot \left(\boldsymbol{E} - \boldsymbol{\Omega}_u \cdot \boldsymbol{\Omega}_u{}^{T}\right) \cdot \boldsymbol{\omega}_0 \cdot \boldsymbol{\omega}_1{}^{T} \cdot \left(\boldsymbol{E} - \boldsymbol{\Omega}_u \cdot \boldsymbol{\Omega}_u{}^{T}\right) \cdot \boldsymbol{\omega}_1} \cdot \cos \gamma \tag{28}$$

and

$$\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_0\, \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_1 =\cdot \sqrt{\left(\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_0\, \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_0\right) \cdot \left(\boldsymbol{\omega}_1{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_1\, \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_1\right)} \cdot \cos \gamma \tag{29}$$

Instead of solving eq. (30) directly for $\boldsymbol{\Omega}_u$ the abbreviations

$$\boldsymbol{x} := \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_0 \tag{30}$$

$$\boldsymbol{y} := \boldsymbol{\Omega}_u{}^{T}\boldsymbol{\omega}_1 \tag{31}$$

for the unknown scalar vector products are to be determined. Then eq. (30) can be written as

$$\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{x} \cdot \boldsymbol{y} =\cdot \sqrt{\left(\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_0 - \mathbf{x}^2\right) \cdot \left(\boldsymbol{\omega}_1{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{y}^2\right)} \cdot \cos \gamma. \tag{32}$$

Allowing only linear changes in the cylindrical coordinates angle γ can be expressed according to eq.(5) and eqs. (31),(32)

$$\gamma = \frac{\boldsymbol{x}+\boldsymbol{y}}{2} \boldsymbol{t}_1. \tag{33}$$

Inserting eq. (34) into (33) yields the non-linear equation to be solved for the unknowns $\mathbf{x}, \boldsymbol{y}$:

$$\frac{\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{x} \cdot \boldsymbol{y}}{\sqrt{\left(\boldsymbol{\omega}_0{}^{T}\boldsymbol{\omega}_0 - \boldsymbol{x}^2\right) \cdot \left(\boldsymbol{\omega}_1{}^{T}\boldsymbol{\omega}_1 - \boldsymbol{y}^2\right)}} =\cdot \cos \left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2} \boldsymbol{t}_1\right) \tag{34}$$

To determine all allowable values for $\mathbf{x}, \boldsymbol{y}$ analytically does not seem to be possible. In order to simplify eq. (35) the value of $\boldsymbol{y}$ is set to zero,

$$\boldsymbol{y} = \mathbf{0}, \tag{35}$$

to yield for eq. (35)

$$\underbrace{\frac{\boldsymbol{\omega_0}^T \boldsymbol{\omega_1}}{\sqrt{(\boldsymbol{\omega_0}^T \boldsymbol{\omega_0} - \boldsymbol{x}^2) \cdot \boldsymbol{\omega_1}^T \boldsymbol{\omega_1}}}}_{:= f_1(x)} =\cdot \underbrace{\boldsymbol{cos}\left(\frac{\boldsymbol{x}}{\mathbf{2}} \cdot \boldsymbol{t_1}\right)}_{:= f_2(x)} \tag{36}$$

Unfortunately, for these kind of equations no analytical solution for **x** is known. However, with the little restriction

$$\boldsymbol{\omega_0}^T \boldsymbol{\omega_1} > \mathbf{0} \tag{37}$$

it is clearly the case that the left-hand-side in eq. (37), function $\boldsymbol{f_1(x)}$, *always* intersects at least once with function $\boldsymbol{f_2(x)}$, the right-hand-side of eq. (37), because:

1. Function $\boldsymbol{f_1(x)}$>0 and the function value is real for $\mathbf{x} < |\boldsymbol{\omega_0}|$
2. The initial value of $\boldsymbol{f_1(x)}$is smaller one, $\boldsymbol{f_1(x = 0) = cos(\alpha) < 1}$ (see Figure 4)
3. The functional value of $\boldsymbol{f_1(x)}$ approaches infinity as $\boldsymbol{x}$ approaches $|\boldsymbol{\omega_0}|$, $\lim_{x \to |\omega_0|} f_1(x) \to +\infty$
4. The initial value of $\boldsymbol{f_2(x)}$, $\boldsymbol{f_2(x = 0)}$, is equal to one and th(38)us larger than $\boldsymbol{f_1(x = 0)}$, i.e. $\boldsymbol{f_1(x = 0) < f_2(x = 0)}$.

Example

In order to demonstrate the solving of eq. (37) the following values have been used:

$$\boldsymbol{\omega_0}^T = \frac{\boldsymbol{\pi}}{\mathbf{180}} \cdot [+\mathbf{2}\ \ +\mathbf{3}\ \ +\mathbf{8}]^T \frac{\boldsymbol{deg}}{\boldsymbol{sec}} \tag{38}$$

$$\boldsymbol{\omega_1}^T = \frac{\boldsymbol{\pi}}{\mathbf{180}} \cdot [-\mathbf{2}\ \ +\mathbf{5}\ \ -\mathbf{1}]^T \frac{\boldsymbol{deg}}{\boldsymbol{sec}} \tag{39}$$

with $\boldsymbol{\omega_0}^T \boldsymbol{\omega_1} > \mathbf{0}$ and a slewing time of $\boldsymbol{t_1} = \mathbf{20}\ \boldsymbol{s}$.
Using MATLAB function fzero [4] to solve for $\boldsymbol{f_1(x) - f_2(x) = 0}$ and initial guess $\boldsymbol{x_0} = \mathbf{0}$ the solution $\boldsymbol{x_s} = \mathbf{0.14101}$ is retained. In Figure 7 the intersection of functions $\boldsymbol{f_1(x)}$ and $\boldsymbol{f_2(x)}$ is visualized for different values of $\boldsymbol{t_1}$ in order to demonstrate that respecting eq. (38) there is always at least one solution to solve eq. (37).

Determination of rotation vector $\boldsymbol{\Omega_u}$

To finalize a unit rotation vector $\boldsymbol{\Omega_u}$ satisfying eqs. (31) and (32) needs to be determined, i.e. $\boldsymbol{\Omega_u}$ needs to satisfy

$$\begin{bmatrix} - \boldsymbol{\omega_0}^T - \\ - \boldsymbol{\omega_1}^T - \end{bmatrix} \cdot \boldsymbol{\Omega_u} = \begin{bmatrix} \boldsymbol{x_s} \\ \mathbf{0} \end{bmatrix} \tag{40}$$

and of course

$$|\boldsymbol{\Omega_u}| = \mathbf{1}. \tag{41}$$

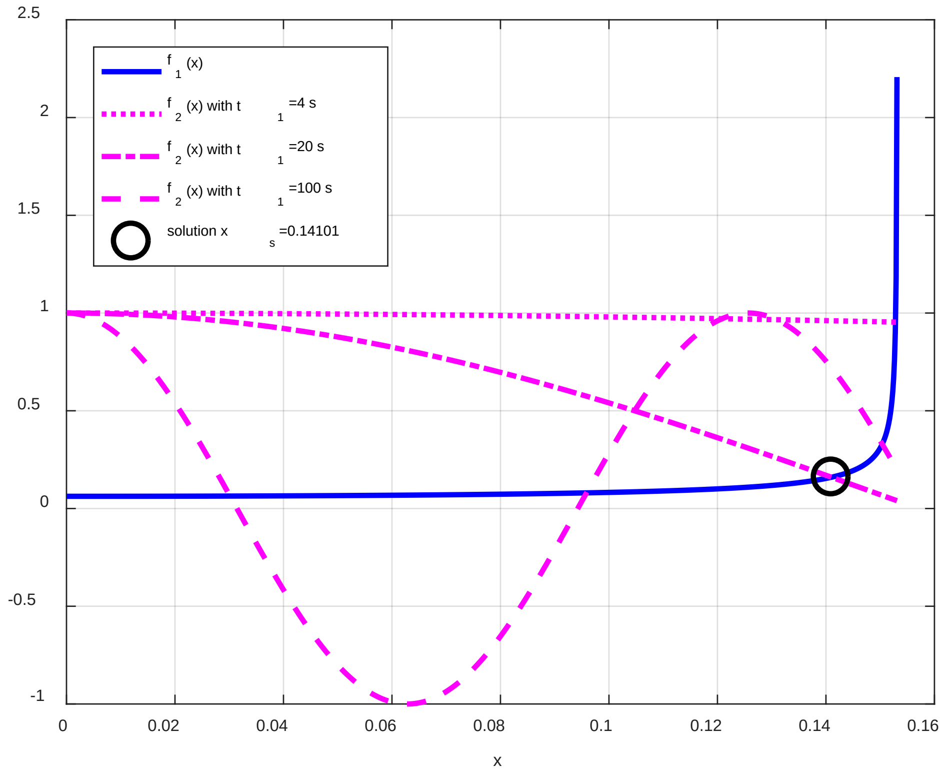


Figure 10: Visualization of the solutions of the nonlinear-equation (37)

The condition for the solution of eqs. (40)-(42) can be visualized by looking at the $\boldsymbol{\omega_0}$-$\boldsymbol{\omega_1}$-plane "from top" in Figure 8: The dash-dot lines there show the possible solutions for any vectors $\boldsymbol{\Omega_u}$ no matter their magnitude, and the intersection of those visualizes the solution. As long as the vectors $\boldsymbol{\omega_0}$ and $\boldsymbol{\omega_1}$ are not parallel this intersection exists; if they are parallel the rate vector is constant in its direction and the attitude determination can be solved analytically as shown in eq. (15).

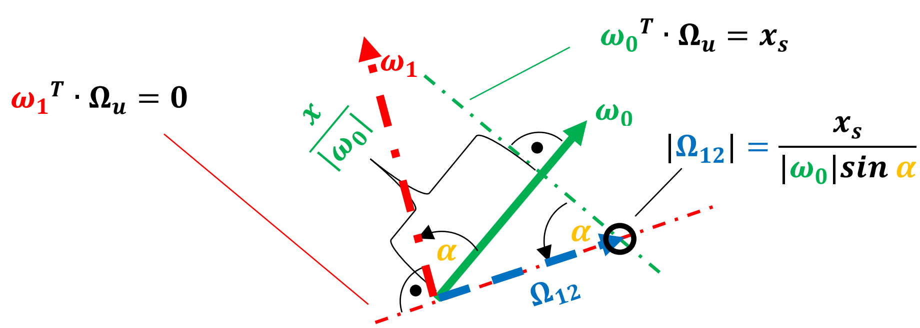


Figure 11: Construction of projection of rotation vector $\boldsymbol{\Omega}$, $\boldsymbol{\Omega_{12}}$, into ω_0-ω_1-plane

The magnitude of the projection of rotation vector $\boldsymbol{\Omega_u}$ within the $\boldsymbol{\omega_0}$-$\boldsymbol{\omega_1}$-plane, $\boldsymbol{\Omega_{12}}$, needs to be smaller than one – otherwise $\boldsymbol{\Omega_u}$ may not have unity magnitude. From triangle trigonometry in Figure 8 the condition

$$|\boldsymbol{\Omega}_{12}| = \frac{\boldsymbol{x_s}}{|\boldsymbol{\omega_0}|\boldsymbol{sin}\ \boldsymbol{\alpha}} < \mathbf{1} \tag{42}$$

needs to hold. It needs to be checked if this condition matches also with the equation to determine $\boldsymbol{x_s}$ in eq. (37). Rearranging (43) yields

$$\begin{gathered}
\boldsymbol{x_s} < |\boldsymbol{\omega_0}|\boldsymbol{sin}\ \boldsymbol{\alpha} \\
\boldsymbol{x_s}^2 < |\boldsymbol{\omega_0}|^2(\boldsymbol{sin}\ \boldsymbol{\alpha})^2 \\
-\boldsymbol{x_s}^2 > -|\boldsymbol{\omega_0}|^2(\boldsymbol{sin}\ \boldsymbol{\alpha})^2 \\
|\boldsymbol{\omega_0}|^2 - \boldsymbol{x_s}^2 > |\boldsymbol{\omega_0}|^2(\mathbf{1} - (\boldsymbol{sin}\ \boldsymbol{\alpha})^2) \\
(\boldsymbol{cos}\ \boldsymbol{\alpha})^2 < \mathbf{1} - \frac{\boldsymbol{x_s}^2}{|\boldsymbol{\omega_0}|^2} \quad .
\end{gathered} \tag{43}$$

From eq. (37) to determine $\boldsymbol{x_s}$ another bound on $(\boldsymbol{cos}\ \boldsymbol{\alpha})^2$ can be derived:

$$\begin{aligned}
\frac{\boldsymbol{\omega_0}^T\boldsymbol{\omega_1}}{|\boldsymbol{\omega_1}|} &= \sqrt{(\boldsymbol{\omega_0}^T\boldsymbol{\omega_0} - \boldsymbol{x_s}^2)} \cdot \boldsymbol{cos}\left(\frac{\boldsymbol{x_s}}{\mathbf{2}} \cdot \boldsymbol{t_1}\right) \\
\boldsymbol{cos}\ \boldsymbol{\alpha} &= \sqrt{\left(\mathbf{1} - \frac{\boldsymbol{x_s}^2}{|\boldsymbol{\omega_0}|^2}\right)} \cdot \boldsymbol{cos}\left(\frac{\boldsymbol{x_s}}{\mathbf{2}} \cdot \boldsymbol{t_1}\right) < \sqrt{\left(\mathbf{1} - \frac{\boldsymbol{x_s}^2}{|\boldsymbol{\omega_0}|^2}\right)} \\
(\boldsymbol{cos}\ \boldsymbol{\alpha})^2 &\qquad\qquad < \mathbf{1} - \frac{\boldsymbol{x_s}^2}{|\boldsymbol{\omega_0}|^2}
\end{aligned} \tag{44}$$

Since eqs. (44) and (45) match exactly for the solution $\boldsymbol{x_s}$ that solves eq. (37) $\boldsymbol{x_s}$ has also the proper value to form a unit rotation vector $\boldsymbol{\Omega_u}$ matching eqs. (31),(32). The unit rotation vector $\boldsymbol{\Omega_u}$ can now be derived from Figure 8 to be

$$\boldsymbol{\Omega_u} = \boldsymbol{\Omega_{12}} + \frac{\widetilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\omega_1}}{|\widetilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\omega_1}|} \cdot \sqrt{\mathbf{1} - \frac{\boldsymbol{x_s}^2}{(\boldsymbol{sin}\ \boldsymbol{\alpha})^2}} \tag{45}$$

with

$$\boldsymbol{\Omega_{12}} = \frac{\widetilde{\boldsymbol{\omega_1}}}{|\boldsymbol{\omega_1}|} \cdot \frac{\widetilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\omega_1}}{|\widetilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\omega_1}|} \cdot \frac{\boldsymbol{x_s}}{\boldsymbol{sin}\ \boldsymbol{\alpha}}. \tag{46}$$

while the "~" corresponds to the matrix formulation of a cross product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. $\boldsymbol{a} \times \boldsymbol{b} =: \widetilde{\boldsymbol{a}} \cdot \boldsymbol{b}$ where

$$\widetilde{\boldsymbol{a}} := \begin{bmatrix} \mathbf{0} & -a_z & +a_y \\ +a_z & \mathbf{0} & -a_x \\ -a_y & +a_x & \mathbf{0} \end{bmatrix} \tag{47}$$

Inserting the values of the example from eqs. (39), (40), the slewing time $\boldsymbol{t_1} = \mathbf{20}\ \boldsymbol{s}$ and the solution $\boldsymbol{x_s}$from Figure 7 into eq. (47) yields

$$\boldsymbol{\Omega_u}^T = [-0.1142 \quad 0.1507 \quad 0.9820]^T \tag{48}$$

SIMULATION RESULTS

Now that the rotation vector $\boldsymbol{\Omega_u}$ is determined (see Figure 9) the rate can be transformed into cylindrical coordinates as shown in Figure 6. Inserting eq. (49), $\boldsymbol{t_1} = \mathbf{20}\ \boldsymbol{s}$ and the solution $\boldsymbol{x_s}$ into eqs. (27), (28) yields

$$\boldsymbol{\omega}_{0r} = [0.0510 \quad 0.0311 \quad 0.0012], \boldsymbol{\omega}_{1r} = [0.0349 \quad 0.0873 \quad -0.0175]. \tag{49}$$

Then further inserting eq. (49), $\boldsymbol{t_1} = \mathbf{20}\, \boldsymbol{s}$ and the solution $\boldsymbol{x_s}$ into eqs.(34) and (5) and use eq. (50) as in Figure 4 yields

$$\gamma(0) = 0, \gamma(t_1) = x_s \frac{t_1}{\mathbf{2}} = 1{,}4101 \tag{50}$$

$$\boldsymbol{a}(0) = |\boldsymbol{\omega}_{0r}| = 0{,}1532, \boldsymbol{a}(t_1) = |\boldsymbol{\omega}_{1r}| = 0{,}0956 \tag{51}$$

$$\Omega_s(0) = 0, \Omega_s(t_1) = 0{,}14101. \tag{52}$$

The values in between those defining lower and upper limits in eqs. (51)-(53) are linearly interpolated and a rate profile in Cartesian coordinates is computed according to eq. (23). The result is shown in Figure 10. From this rate profile the attitude transformation matrix is computed in three ways:

1. Analytically as in eq. (22) (here $T_{b\,0}^{i} = E$)
2. Numerically with MATLAB-function ode45, see [4]
3. Numerically with MATLAB-function ode113, see [4]

The results are transformed into Euler angles in sequence “1-2-3” and is shown in Figure 11. In there, as an error-measure the magnitude of the numerical transformation against the inverse analytical transformation is given which shows a very tiny error w.r.t. the analytical solution, maximal 10^{-10} deg. The largest error value over time is taken as an accuracy measure for further experiments.

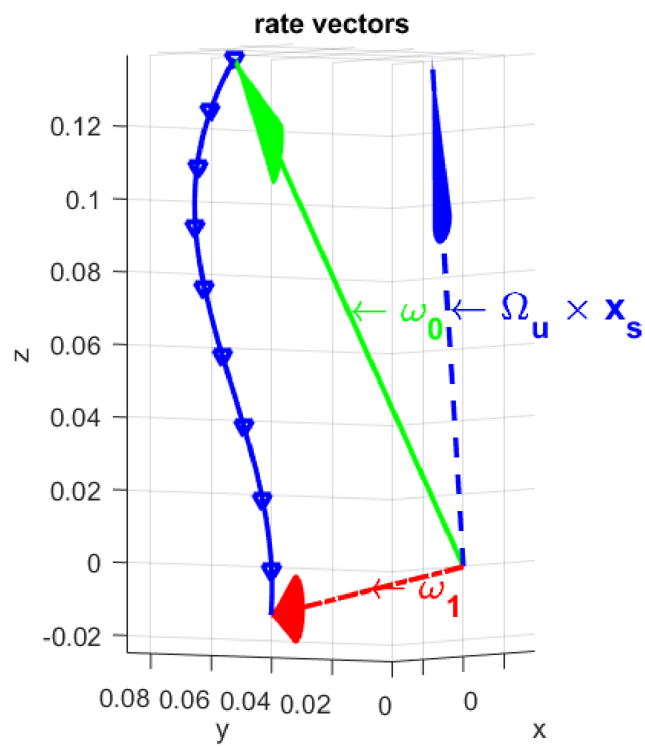


Figure 12: Progression of rates from initial rate ω_0 until final rate ω_1

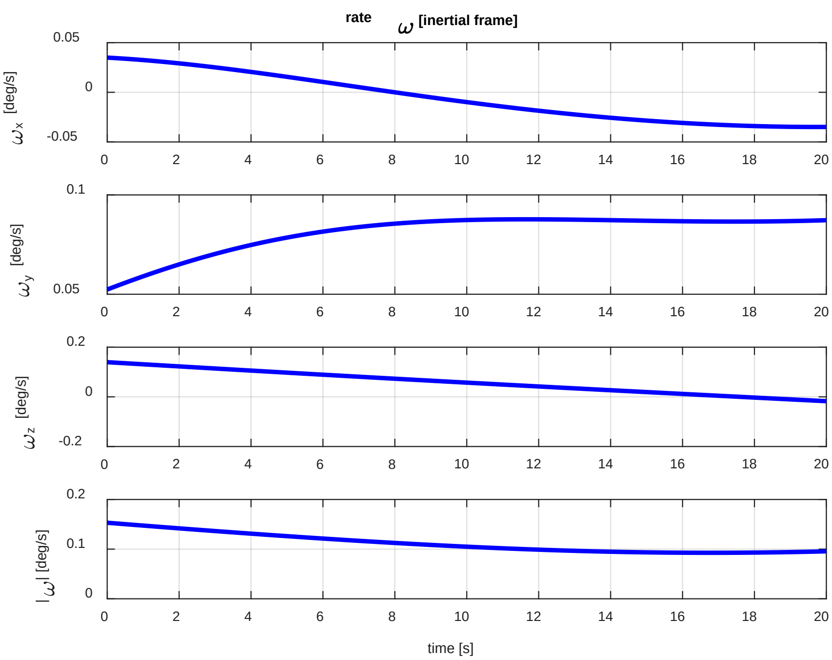


Figure 13: Rate progression of the simulation example

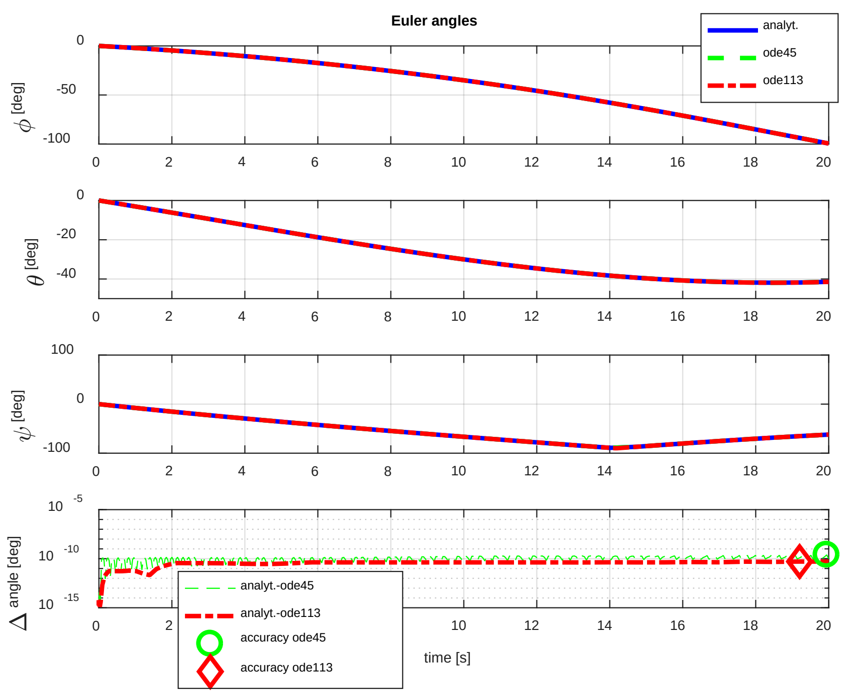


Figure 14: Attitude-progression of the simulation example

Summary of a complete simulation campaign

In order to investigate the computation time (using a modern Intel core i7 processor) and accuracy benefits of the proposed method the results for slew times between 0,1 s and 100 s are displayed in Figure 12 and Figure 13. It shows that the computation time of the analytical solution is by a factor of 100 faster than the numerical solutions.

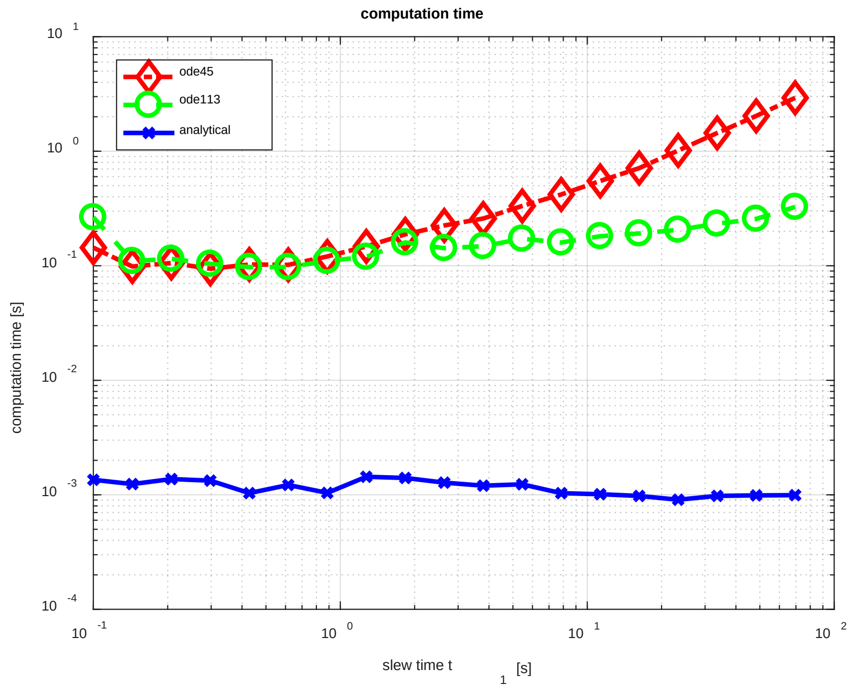


Figure 15: Computation time needed for analytical and numerical simulation

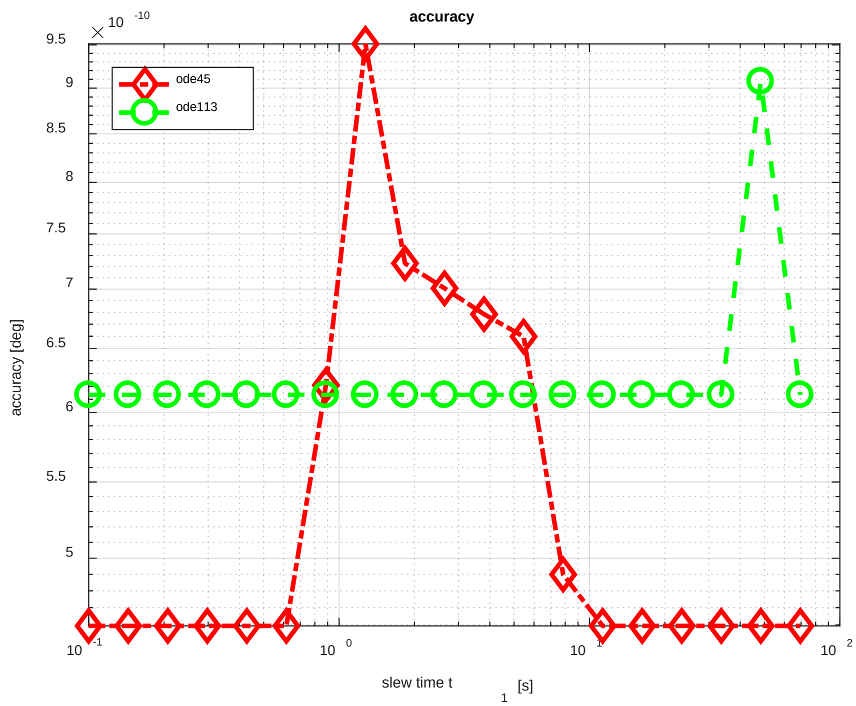


Figure 16: Accuracy received for numerical simulations

CONCLUSIONS

A method to compute the attitude evolution from specific slewing rate profiles is presented. For theses profiles the kinematic differential equation can be solved analytically. It avoids time consuming numerical solving of the kinematic differential equation and is about a 100 times faster. The method is transparent a needs basic mathematical concepts. It can be used for optimizations of rate and/or attitude behavior and for the planning of corresponding trajectories.

REFERENCES

[1] Patera, Russel P., *Attitude Propagation for a Slewing Rate Vector,* AIAA Guidance, Navigation and Control Conference, 10-13 August 2009, Chicago, Illinois.

[2] Bong Wie, *Space Vehicle Dynamics and Control,* AIAA education series, page 340, equation 5.59, 2008.

[3] Zentgraf, Peter, *Private Communications “Determine Possible Rotations to Bring a Vector Pair into a Suitable Attitude”*, 24.12.2015.

[4] www.mathworks.com