Proceedings of 3rd
European Control Conference
Rome, Italy, September 1995

A Frequency Domain Method for Performance-Robust Control System Design

Peter Zentgraf and Georg Grübel
DLR - Oberpfaffenhofen
Institute for Robotics and System Dynamics
Control Design Engineering
D-82234 Wessling, Germany
df12@master.df.op.dlr.de

Keywords: Uncertainty, Performance robustness, Controller synthesis, Controller analysis

Abstract

A robust-controller synthesis and analysis method which works in the frequency domain and has similarity with the QFT-method of Horowitz is presented. Plant uncertainty is described for specific frequencies as bounded two dimensional regions within the complex plane in the design process. The design goal is to locate the magnitude of the frequency responses of the feedback system within predetermined bounds. Newly developed substitute design conditions simplify the determination of appropriate controller transfer functions. Alternatively, for an already given controller the sharpest possible bounds on the magnitude of uncertain feedback system frequency responses can be determined.

1 Introduction

Most control design methods refer to a "nominal" design model of the plant. The underlying assumptions are that the design model describes the behaviour of the plant with sufficient accuracy and that the plant dynamics do not change in operation. Problems may arise due to the modelling/control-design inseparability problem provoking the fact that a controller suitably designed on base of a nominal design model may deteriorate the performance of the physical closed loop system up to instability. Peterson [7] has reported a practical example: For vibration control of a truss structure, a controller had been designed according to the theory of optimal projection. This controller destabilized the real system, because the first natural frequency of the finite-element-model had not been modeled accurately enough.

Various concepts have been developed to take into account the effects of uncertainty in plant modelling for control purposes. These are the concepts of parameter insensitive control, of robust control and of generic stability robustness by hyperstability control.

Since signal feedback always impacts system stability, which means that a feedback control system with a given controller may be destabilized by changing the plant dynamics, stability is the fundamental issue in the presence of big model uncertainties. Therefore, most publications in analysis and synthesis of robust control systems refer to "stability robustness". The problem field can be extended by transferring the issue robustness from the qualitative term "stability" to the quantitative term "performance". A step into this direction was the introduction of the term "stability performance robustness", which has been treated in detail in the book of Ackermann [1]. But good performance in general requires more than good stability performance, in that tracking and disturbance rejection properties need to be evaluated in addition to stability. Therefore, the term "performance robustness" is introduced: Assume the controller design method takes explicitly into account modelling uncertainties described as a bounded plant model family which contains some plant model defined as the "nominal" model. Then the closed loop system is considered as "performance robust", if it is stable for every plant family member *and* if compared to the nominal performance a certain minimal performance of tracking or disturbance rejection is guaranteed.

The contribution of this paper is a method to achieve performance robustness in the sense defined above. Especially for the control of flexible structures, structural dynamic parameters and control performance can be put into direct context.

Core of the method is a theorem from functional analysis, the so-called Schauder fixed point theorem. Thoughtful application of this theorem enables the control engineer controller synthesis as well as controller analysis to be done on a substitute control system based on a nominal model, where parametric uncertainty enters as an upperbounded disturbance. This is different to the MIMO (multiple input/multiple output)-QFT method of Horowitz [4] where a kind of substitute control design model is used which still depends on uncertain system parameters. Once the substitute disturbance is suitably upper bounded, the newly developed method allows the design for SISO (single input/single output) as well as for MIMO (multiple input/multiple output) systems in a transparent way.

Design specifications are formulated as bounding functions on certain transfer functions of the closed loop system. While this paper concentrates on performance robust tracking, it is also possible to apply the method on performance robust disturbance rejection including control effort.

2 Performance Robustness Conditions

The method works entirely in the frequency domain.

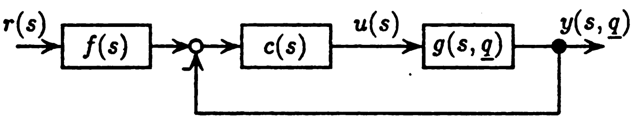

Figure 1: Standard Control Loop System.

Consider the control loop in figure 1. The uncertain plant is represented by the transfer function $g(s,\underline{q})$. The uncertain parameters q_i are assumed to be uncertain but bounded and comprised in the vector $\underline{q}$. The plant input signal u is the controller output signal, the plant output signal is y, the reference signal is r. Systems to be designed are the prefilter transfer function $f(s)$ and the controller transfer function $c(s)$. They shall be designed in such a way that the output signal y follows the reference signal r "as good as possible".

The tracking transfer function $t(s,\underline{q})$ describes the dynamic relation between $r(s)$ and $y(s,\underline{q})$:

$$y(s,\underline{q}) = t(s,\underline{q})r(s), \tag{1}$$

which can be derived from figure 1 as

$$t(s,\underline{q}) = f(s)\frac{g(s,\underline{q})c(s)}{1+g(s,\underline{q})c(s)}. \tag{2}$$

Ideally, the magnitude of the tracking transfer function should be made unity, since then from (1) $y = r$. This is not realisable, because for large frequencies the signal gain of real systems goes to zero. Hince this tracking requirement can be approximated only up to a certain frequency within the bandwidth of the system.

Tracking behaviour of the closed loop system is considered as performance robust if the following "performance robustness conditions" are satisfied:

1. The closed loop system is stable for all uncertain but bounded parameters $\underline{q}$.
2. The magnitude of the tracking transfer function $t(s,\underline{q})$ is bounded for all uncertain but bounded parameters $\underline{q}$:

   $$a(\omega) \leq |t(j\omega,\underline{q})| \leq b(\omega) \quad \forall\omega,\underline{q}. \tag{3}$$

   The bounding functions $a(\omega)$ and $b(\omega)$ are a priori specified and are assumed to be real and continuous.

Two different problems can be considered to achieve performance robustness:

1. To solve the controller synthesis problem an appropriate controller and an appropriate prefilter transfer function need to be designed.

   In general a systematic solution of the problem directly based on (3) is necessarily iterative and hence quite tedious, because stability and magnitude bounding can not be treated separately. A natural solution were to guess a prefilter and a controller first, then to check the closed loop system for stability for all parameters $\underline{q}$ and the validity of (3). After appropriate changes of controller and prefilter parameters the iteration loop could start again. This way of doing is not a transparent design procedure. In the sequel we use instead a separation approach based on solving appropriate "substitute design conditions".
2. To solve the controller analysis problem, bounding functions $a(\omega)$ and $b(\omega)$ need to be determined for a given controller and prefilter. The bounding functions $a(\omega)$ and $b(\omega)$ should have minimum distance to the uncertain magnitude $|t(j\omega,\underline{q})|$. For instance in figure 2 the bounding functions $a_1(\omega)$ and $b_1(\omega)$ are preferable to $a_2(\omega)$ and $b_2(\omega)$, since the first ones are less conservative than the second ones.

3 Substitute Design Conditions

A design based on performance robustness conditions (3) directly can be avoided. The idea is to map the uncertain original control system into a substitute nominal control system which does not contain the parametric uncertainty itself, rather the uncertainty is taken care of by the mapping.

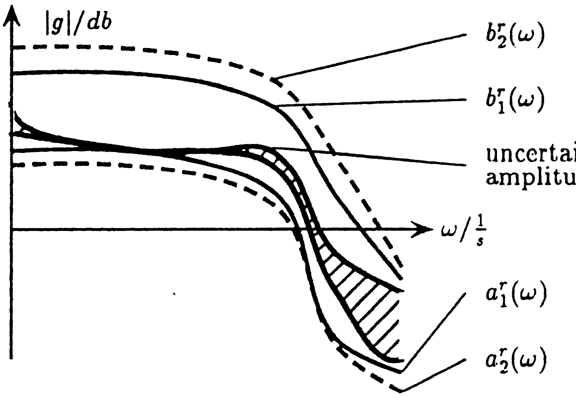

Figure 2: Example for Appropriate Bounding Functions.

The tracking transfer function is rearranged from an explicit form into an implicit form. From (2) the following equation can be deduced:

$$t(s,\underline{q}) = \frac{f(s)c(s)p(s) + (1 - \frac{p(s)}{g(s,\underline{q})})t(s,\underline{q})}{1 + c(s)p(s)}. \quad (4)$$

Denoting the term $(1 - \frac{p}{g})t$ with d_r, equation (4) can be interpreted as a control system shown in figure 3. In this substitute control system the substitute plant transfer function $p(s)$ which does not depend on system parameters $\underline{q}$ and can freely be selected with the same pole excess as the original plant transfer function. Uncertainty enters the system from outside as a substitute disturbance d_r.

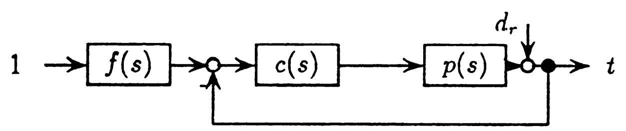

Figure 3: Substitute Control System.

So far nothing is gained with this problem reformulation, because the substitute disturbance d_r depends on the unknown tracking transfer function t which itself is to be designed. Hence d_r can not be bounded without knowing the controller transfer function. But: if the Schauder fixed point theorem is applicable, it allows this vicious circle to be split up. Then the substitute disturbance d_r can be bounded and the closed loop system can be stabilized for all uncertain system parameters $\underline{q}$.

Schauder's fixed-point theorem states [3]: "A continuous operator mapping a closed, convex set of a banach space into itself has a fixed point, if the mapped set is compact." Since in a Banach space a compact set is also a closed set and since a continuous mapping of a compact set is again compact [6] said theorem can be formulated as: "A continuous operator mapping a compact, convex set of a banach space into itself has a fixed point."

The idea to apply this theorem to robust control problems was first developed by Horowitz. He recognized the implicit formulation of the tracking transfer function being useful for defining a fixed point of an appropriate function. In [4] Horowitz then formulated a kind of substitute control problem formulation for MIMO-systems, only to reduce the design of a system with n inputs and n outputs to the independent design of n^2 SISO systems. There, the uncertain system parameters $\underline{q}$ are still within the closed loop, whereas in our approach the uncertainty is extracted and mapped in a substitute disturbance d_r.

The details of the application of Schauder's fixed-point theorem are omitted; the interested reader may study them in [4, 8]. As a result, instead of solving the performance robustness conditions directly, the following two inequalities need to be satisfied for all ω:

$$a(\omega) < \left|\frac{c(j\omega)p(j\omega)f(j\omega)}{1 + c(j\omega)p(j\omega)}\right| - \left|\frac{r(\omega)}{1 + c(j\omega)p(j\omega)}\right| \quad \forall\omega \quad (5)$$

$$b(\omega) > \left|\frac{c(j\omega)p(j\omega)f(j\omega)}{1 + c(j\omega)p(j\omega)}\right| + \left|\frac{r(\omega)}{1 + c(j\omega)p(j\omega)}\right| \quad \forall\omega. \quad (6)$$

The substitute disturbance d_r is upperbounded by

$$|d_r(j\omega)| \le \max_{\underline{q}} \left\{ \left|1 - \frac{p(j\omega)}{g(j\omega,\underline{q})}\right| \right\} b(\omega) =: r(\omega), \quad (7)$$

and a way to derive bounding functions $r(\omega)$ on $d_r(\omega)$ is shown in the next section. Furthermore, to guarantee robust stability of the feedback system, it is required that

1. transfer functions $f(s)$, $p(s)$ and $\frac{c(s)p(s)}{1+c(s)p(s)}$ are stable and
2. plant transfer function $g(s,\underline{q})$ has for all $\underline{q}$ no zero in the right open complex half plane.

Inequalities (5) and (6) and restrictions addressing the pole-zero-configuration of the involved transfer functions are referred to as "substitute design conditions".

Before discussing design procedures for the unknown control transfer function $c(s)$ and for the unknown substitute plant transfer function $p(s)$, the next section presents possible bounding functions $r(\omega)$ for the substitute disturbance d_r in equation (7).

4 Substitute Disturbance Bounds

To get a bounding function for the substitute disturbance d_r the term $\max_{\underline{q}} \left\{ \left|1 - \frac{p(j\omega)}{g(j\omega,\underline{q})}\right| \right\}$ in equation (7) must be bounded. At a fixed frequency ω this can be achieved by mapping the uncertain but bounded box of system parameter $\underline{q}$, which is in general multi-dimensional, to the value set of the plant frequency response, which is a bounded two-dimensional region in the complex plane as shown in figure 4.

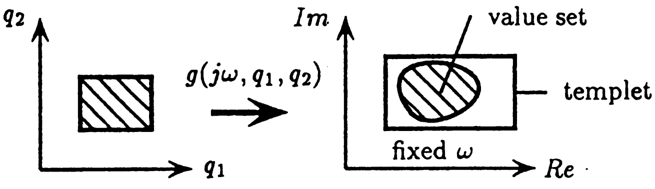

Figure 4: Parameter Uncertainty Mapping.

In the design process only the two-dimensional value set is considered when taking system uncertainty into account. In the book of Ackermann et. al. [1] numerical techniques to construct these value sets are described. But often it is not necessary to know the exact value set and an approximation can be derived which is suitable for design but requires less computational effort. Such an approximation is called a "templet".

Knowing a system transfer function analytically with the uncertain parameters entering polynomially, a simple way to construct these templets can be developed. Putting $s = j\omega$, both in the numerator- and in the denominator polynomial all positive and all negative terms of the real and the imaginary part are grouped together. Substituting in these groups of the system parameters the upper bounds and the lower bounds, respectively, yields rectangular templets for the numerator- as well as the denominator polynomial. These templets can be used to bound the value set of the system frequency response. Furthermore, this procedure has the advantage that it leads to templets which yield guaranteed bounds of the value set of the frequency response. On the other hand, the use of templets may lead to an overestimation of system uncertainty.

Nevertheless, the use of templets has two advantages: First, the system parameters can enter in an arbitrary way the transfer function, i.e. linear dependency of the parameters is not required. This is especially useful for the control of flexible structures where the system parameter dependency can be highly nonlinear [8]. Secondly, templets can be used for bounding the substitute disturbance according to equation (7). In there, the term $1-\frac{p(j\omega)}{g(j\omega,\underline{q})}$ can be considered as a function which maps at a fixed frequency ω the templet for $g(j\omega,\underline{q})$ to a value set for which another templet can be constructed [8]. Then, the problem to upper bound the substitute disturbance d_r, i.e. to solve $\max_{\underline{q}}\left\{\left|1-\frac{p(j\omega)}{g(j\omega,\underline{q})}\right|\right\}$, is equal to finding that value in the value set for $1-\frac{p(j\omega)}{g(j\omega,\underline{q})}$ which has the maximum distance to the origin. An upper bound can be found easily by determining the maximum distance of the corners of this templet to the origin.

5 Design Procedures

The substitute design condition inequalities (5 - 6) must be satisfied within the whole continuous frequency spectrum from $\omega = 0$ until $\omega \to \infty$. In practice, the conditions are checked at finite and discrete test-frequencies only. It is assumed that by careful selection of the test-frequencies the conditions for the actual continuous spectrum are also satisfied. Careful selection means that those frequencies are to be chosen which yield "characteristics" of the frequency response. For instance, for the control of flexible structures the test-frequencies should include the undamped natural frequencies. In order to guarantee that the conditions are satisfied for $\omega \to \infty$ from inequalities (5 - 6) criteria for the gain factors of the involved transfer functions can be derived [8].

To solve the controller synthesis problem, stability of $\frac{c(s)p(s)}{1+c(s)p(s)}$ is required. Therefore, it seems feasible to design the transfer function

$$\bar{c}(s) := \frac{c(s)p(s)}{1+c(s)p(s)} \tag{8}$$

and to consider stability of $\bar{c}(s)$ and $p(s)$.

It will be shown that by introducing the design transfer function $\bar{c}(s)$, also another advantage is obtained in that each design transfer function $\bar{c}(s)$ and $p(s)$ allows to manipulate the left and the right side of the inequality condition independently.

The independent transfer functions $\bar{c}(s)$ and $p(s)$ are now used as free design transfer functions to fulfill the substitute design conditions. To solve the controller synthesis problem inequalities (5 - 6) have to be brought into appropriate form yet, where the controller transfer function $c(s)$ and the prefilter transfer function $f(s)$ do not enter in both inequalities at the same time: First equation (8) is substituted in inequalities (5 - 6) to yield

$$a(\omega) \;<\; |\bar{c}(j\omega)f(j\omega)| \;-\; |1-\bar{c}(j\omega)|\,r(\omega) \;\; \forall\omega \tag{9}$$

$$b(\omega) \;>\; |\bar{c}(j\omega)f(j\omega)| \;+\; |1-\bar{c}(j\omega)|\,r(\omega) \;\; \forall\omega. \tag{10}$$

Then, inequality (9) is multiplied by -1 and added to inequality (10), resulting in

$$|\bar{c}(j\omega)-1|\,r(\omega) \le \frac{b(\omega)-a(\omega)}{2} \;\; \forall\omega. \tag{11}$$

For the design of the prefilter transfer function $f(s)$ a parameter ε is introduced to sharpen inequality (11) so that this inequality (11) becomes

$$|\bar{c}(j\omega) - 1| \leq \frac{b(\omega) - a(\omega)}{2r(\omega)}(1-\varepsilon) \quad \forall\omega \tag{12}$$

with

$$0 \leq \varepsilon < 1. \tag{13}$$

For a fixed frequency ω in the complex $\bar{c}(j\omega)$-plane inequality (12) can be interpreted as a circle about center $-1 + j0$ and with radius

$$R(\omega) := \frac{b(\omega) - a(\omega)}{2r(\omega)}(1-\varepsilon). \tag{14}$$

Admissible circles for robust disturbance rejection and its related control effort can be derived to yield a whole family of circles [8], wherein solutions for $\bar{c}(j\omega)$ can be searched. Their radia are determined by the plant templets, by the design transfer function $p(s)$, and by the performance requirements bounds $a(\omega)$ and $b(\omega)$ only, but not by the transfer functions to be designed.

Once the design transfer function $\bar{c}(s)$ is determined, the prefilter transfer function $f(s)$ can be designed easily as shown in [8].

6 Example

The method is demonstrated by a simple example. Consider the control loop system in figure 1. The plant transfer function is denoted by $g(s,\omega_1,\xi_1)$ replacing $g(s)$ in figure 1 which is in this case a second order lightly damped system:

$$g(s,\omega_1,\xi_1) = \frac{1}{s^2 + 2s\omega_1\xi_1 + \omega_1^2}, \tag{15}$$

where ω_1 is the uncertain natural frequency and ξ_1 is the uncertain damping factor. Only the parameter interval bounds are assumed to be known:

$$\begin{array}{lllllll} \underline{\omega}_1 & := & 0.8s^{-1} & \leq & \omega_1 & \leq & 1.2s^{-1} & =: & \bar{\omega}_1 \\ \underline{\xi}_1 & := & 0.004 & \leq & \xi_1 & \leq & 0.006 & =: & \bar{\xi}_1. \end{array} \tag{16}$$

This corresponds to a parameter uncertainty of 20% about the nominal values $\omega_{10} = 1s^{-1}$ and $\xi_{10} = 0.005$. The natural frequency ω_1 enters nonlinearly the plant transfer function. Figure 5 shows a parameter study of a Bode plot of $g(s,\omega_1,\xi_1)$ for various values of ω_1,ξ_1.

In what follows we want to focus on the controller analysis problem: For a given controller and prefilter transfer function,

$$c(s) = 400\frac{(s+1)^2}{s(s+40)}, \tag{17}$$

$$f(s) = 25\frac{(s+20)^2}{(s+10)^4}, \tag{18}$$

the sharpest possible upper and lower bounding functions $a^r(\omega)$ and $b^r(\omega)$ of all possible magnitudes (depending on ω_1,ξ_1) of the closed loop transfer function are to be determined. The frequency response of the nominal system suggests bounding functions as

$$b(\omega) = \frac{K_b}{|j\omega+10|^4} \quad \text{and} \quad a(\omega) = \frac{K_a}{|j\omega+10|^4}, \tag{19}$$

where K_a and K_b are to be chosen as large and as small as possible, respectively, such that for all ω_1, ξ_1

$$\frac{K_a}{|j\omega+10|^4} \leq |t(j\omega,\omega_1,\xi_1)| \leq \frac{K_b}{|j\omega+10|^4}. \tag{20}$$

The problem shall be solved using the substitute design conditions. The restrictions on the pole-zero-configuration of the involved transfer functions $g(s)$, $p(s)$, $f(s)$ and $\frac{c(s)p(s)}{1+c(s)p(s)}$ on page 3 are satisfied. A substitute plant transfer function $p(s)$ having the same pole excess as $g(s,\omega_1,\xi_1)$ in (15) has been chosen to be $\frac{1}{(s+1)^2}$. If the plant has with very little uncertainty, for $p(s)$ the nominal plant transfer function should be selected, because inequality (12) with reference to (7) can easily be satisfied. Substituting all known equations in the substitute design condition inequality (12) yields

$$\left|\frac{400}{-w^2+40j\omega+400} - 1\right| \leq \frac{K_b - K_a}{2|j\omega+10|^4 r_1(\omega)} =: R(\omega). \tag{21}$$

The function $r_1(\omega)$ upper bounds the term $\max_{\omega_1,\xi_1}\left\{\left|1 - \frac{p(j\omega)}{g(j\omega,\omega_1,\xi_1)}\right|\right\}$ in equation (7) [8] which is a part of the bounding function of the substitute disturbance:

$$r_1(\omega) = \max_{\omega_r,\xi_r}\{\sqrt{\frac{(K_p\omega_r^2 - p_0 - \omega^2(K_p-1))^2 + (\omega(2\omega_r\xi_r - 1))^2}{(-\omega^2+p_0)^2 + (2\omega p_1)^2}} \tag{22}$$

where ω_r and ξ_r can be the upper or lower interval bounds from equation (16):

$$\omega_r \in \{0.8, 1.2\}, \xi_r \in \{0.004, 0.006\}. \tag{23}$$

Next, optimal values for K_a and K_b are determined with the help of software design environment ANDECS_MOPS [5], which has a parameter optimization- and parameter study features well suited for this purpose. Inequality (21) is computed and evaluated for some discrete frequencies between $0.1s^{-1}$ and $100s^{-1}$, leading to the values

$$K_b = 12500 \tag{24}$$

$$K_a = 7500. \tag{25}$$

In figure 6 the result of the design process is visualized. On the left side of figure 6 the satisfaction of the substitute design condition inequality (21) is demonstrated. On the right side, the satisfaction of the performance robustness condition inequality (20) is confirmed by a parameter study where the frequency response of the closed loop system is computed for various values of the uncertain system parameters ω_1 and ξ_1.

7 Conclusions

A frequency domain method has been proposed to guarantee robust performance in the presence of system uncertainties which are described as templets at certain frequencies. The method enables to easily incorporate closed-loop stability by satisfying pole/zero location restrictions of the involved transfer functions. Then, it is possible to satisfy performance requirements, expressed as frequency response magnitude bounding functions, by considering corresponding substitute design conditions which can be satisfied in a transparent way. The method can also be applied to disturbance rejection problems including the related control effort. For MIMO systems the method leads to controller transfer matrices with diagonal elements [8].

References

[1] Ackermann, J. *Robust Control.* Springer Verlag, 1993.

[2] Bals, J. Parameter Optimized Hyperstable Controllers. In *Proceedings of the 30th Conference on Decision and Control*, Brighton (UK), 1991.

[3] Collatz, Lothar. *Funktionalanalysis und numerische Mathematik.* Springer, Berlin, 1968.

[4] Horowitz, Isaac. Quantitative Synthesis of Uncertain Multiple Input-Output Feedback Systems. *International Journal of Control*, 30(1):81–106, 1979.

[5] Joos, H.-D. *Informationstechnische Behandlung des mehrzieligen regelungstechnischen Entwurfs.* Dr.-Ing. thesis, Universität Stuttgart, 1993.

[6] Kolmogorov, A.N., and Fomin, S.V. *Reelle Funktionen und Funktionalanalysis.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.

[7] Peterson, L.D. *Experimental Verification of Structural Control*, pages 383–396. Computer Mechanical Publications. Springer-Verlag, 1990.

[8] Zentgraf, P. *Anwendung des Schauderschen Fixpunktsatzes zur güterobusten Regelung flexibler Strukturen mit unsicheren Parametern.* Dr.-Ing. thesis, Ruhr-Universität Bochum, 1994.

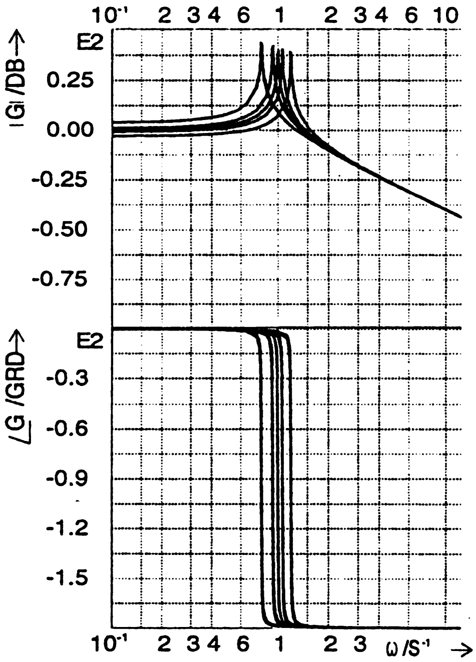

Figure 5: Bode-plot of the uncertain plant transfer function $g(s, \omega_1, \xi_1)$.

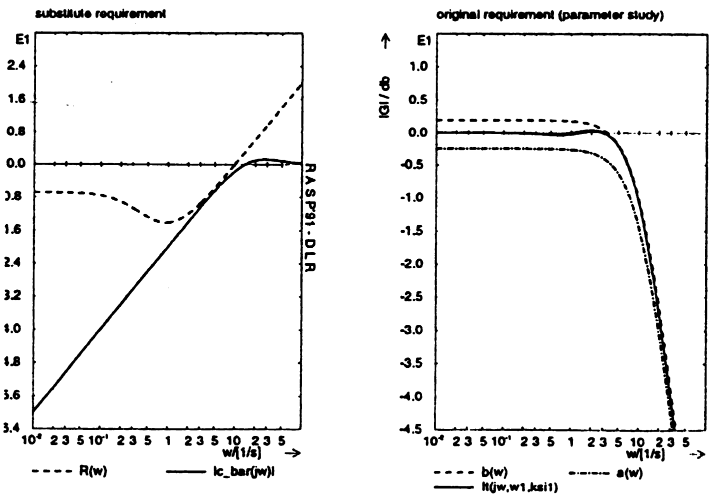

Figure 6: Substitute Design- and Performance Robustness Condition.