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Non-Rest to non-Rest Reference Slews for Agile Imaging Satellites With LQ-minimized Angular Momentum

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This paper shows a full parametrization non-rest to non-rest slew maneuvers using polynomials. A new modelling formulation is developed which on one hand satisfies exactly the given boundary conditions as well as the kinematic differential equation of the rotating body exactly. The idea is to determine one part of the unknown coefficients by the kinematical boundary conditions and to use the other part as free design parameters to shape the dynamic in between the boundary conditions. In addition, a least squares problem is formulated in order to minimize the angular momentum of the slew. This approach allows optimal slew maneuvers as well as low computational need such that an onboard usage is possible.

I.Nomenclature

J = mass moment of inertia matrix of rigid body in body-fixed frame, [kg_ m2]
ω_{bi}^{i} = rotation of the body frame wrt inertial frame and expressed in the inertial frame, [rad/s]
τ^{b} = torque expressed in body frame, [Nm]
h^{b} = angular momentum expressed in body frame, [Nm]
t = time, [s]
CMG = control momentum gyros
T_{b}^{i} = transformation matrix from the body-frame into the inertial frame
v_1, v_2, v_3 *x,y,z*-direction unit vector from inertial-frame into the body frame

II. Introduction

During missions of Earth observation satellites one important task is to scan predefined spots on the surface as shown in **Fig. 1**. For that purpose the satellite needs to slew from a particular initial attitude, an initial rate velocity and a rate acceleration to a final attitude, a final rate velocity and a final rate acceleration [1] for a priori given slew time, in order to perform optimal shots with the high precision camera onboard. Attitude control shall be performed by CMGs.

The slews treated here fall in the category of non-rest to non-rest slews. During the search of a suitable profile the boundary conditions need to be satisfied as well as a coupled linear matrix differential equation, Eq. (4), needs to be solved. There is no general *analytical* solution available for it. This implies numerical integration and thus time consuming computations. This is not what you want during an optimization process. In [1] an analytical solution has been derived for a subset of all possible rate profiles. Therefore, optimality cannot be achieved by this solution in general.

This problem is complicated by the fact that the boundary conditions depend on the slew time itself. Due to the iterative nature of an overall optimization of the maneuver planning it is desired to find feasible slews which require a very low computation time. This is one focus of the presented article.

In [2] non-rest constraints are considered while the focus is laid on the maximization of a singularity measure on the usage of CMGs. The advantage of CMGs is that in principle high torques are available outside of so called “singularities”, i.e. certain geometrical constellations of the CMGs when no torque can be generated. In order to

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minimize the probability of the occurrence of those it is beneficial to use CMGs for short period of times only. Another challenge from an intensive usage results in micro vibrations as reported in [3], because CMGs may influence the performance of high-sensitivity instruments on-board.

The possible drawback on this a bit more cautious usage of CMGs is that a global minimal slew time is unlikely to reach, because in [4] is shown that for rest-to-rest slews the optimal control signal has a bang-bang structure.

Therefore, the second focus is to generate slew profiles which naturally avoid torque signals which require maximal torque for a long period of time.

The scope of this article is to derive explicit reference profiles for slews of agile satellites. A reference profile is understood as a specific time depending set of torque commands which cause optimal behaviour on the attitude, rate, rate acceleration and other potential signals. The corresponding control problem is not treated.

In detail, all slews shall meet predefined boundary conditions on the dynamic state as well as constraints on the commanded input signal:

1. Starting from an arbitrary *attitude* T_0 at slew start the target attitude T_1 shall be accomplished within the given slew time t_1.
2. Starting from an arbitrary *rate* ω_0 at slew start the target rate ω_1 shall be accomplished within the slew time t_1.
3. Starting from an arbitrary *rate acceleration* $\dot{\omega}_0$ at slew start the target rate acceleration $\dot{\omega}_1$ shall be accomplished within the slew time t_1.

While taking these constraints into consideration, the amount of angular momentum during the slew shall be minimized.

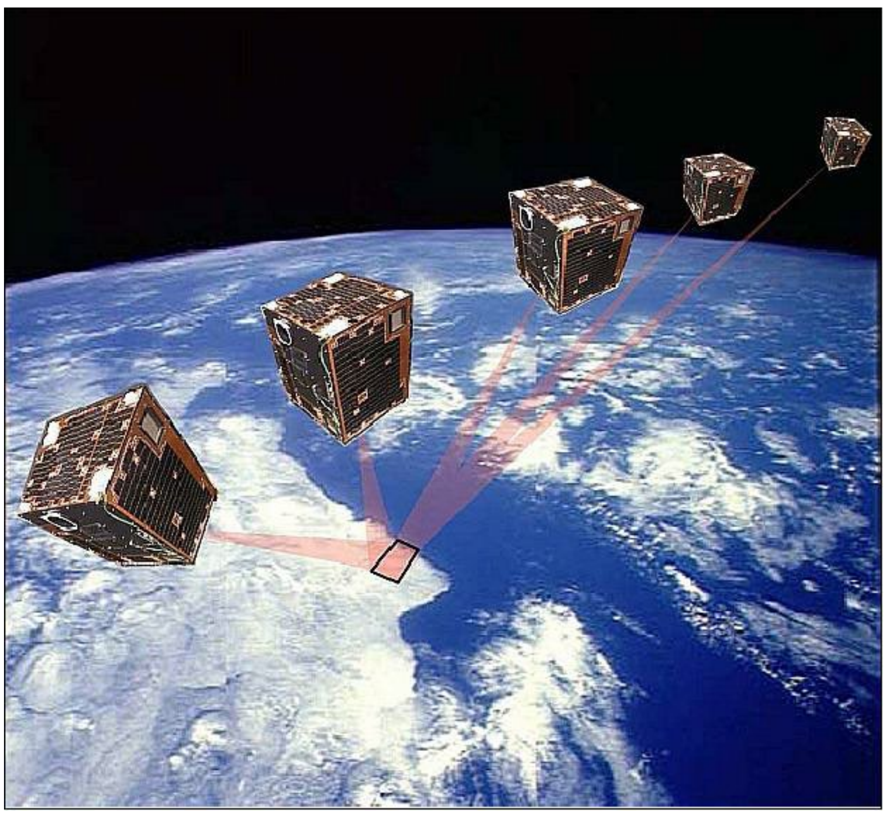

Fig. 1 Illustration of a multi-angle observation sequence of PROBA-1 (image from ESA EO-directory web-portal).

A. Main Contributions

The main contribution of this paper is a full parametrization of non-rest to non-rest slew maneuvers by in total six time dependent polynomials, two orientation vectors of the satellite are described by three polynomials each; clearly, this case includes rest-to-rest maneuvers as well, because the rate velocity can be selected to be zero and the rate acceleration constraint can be omitted. A new modelling formulation is developed which on one hand satisfies exactly

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the given boundary conditions as well as the kinematic differential equation of the rotating body and on the other hand gives an analytical expression of the corresponding rate. For a given freely selectable order of the polynomials the general solution of the slew problem is derived.

In addition, a least squares problem is formulated in order to minimize the angular momentum of the slew. This approach allows optimal slew maneuvers as well as low computational need such that onboard usage is possible.

B. Organization of Paper

In section C the general idea of the parametrization of the slew dynamics in two separated vector descriptions is proposed. Afterwards in section D the given kinematic boundary conditions are translated into the vector boundary conditions. Section E demonstrates a general solution for the mapped vector oriented boundary condition. Section IV describes the developed Least Squares approach and the paper concludes with a numerical example to demonstrate the correctness of the approach.

III.Parametrization with polynomials satisfying boundary conditions

The idea of this section is to propose a parametrization which allows the optimizer to freely select tuning parameters while guaranteeing at the same time that all boundary conditions are satisfied.

1. Derivation of the kinematic differential equation

Let us now consider the change of a vector $\boldsymbol{r(t)}$ – which can be imagined as fixed to a body - during a small time period $\boldsymbol{\Delta t}$ within the inertial frame $\boldsymbol{i}$ due to the rotation around vector $\boldsymbol{\omega_{bi}^{i}(t)}$ – the index $\boldsymbol{i}$ is omitted - as shown in Fig. 1, which leads to the differential equation Eq. (1),

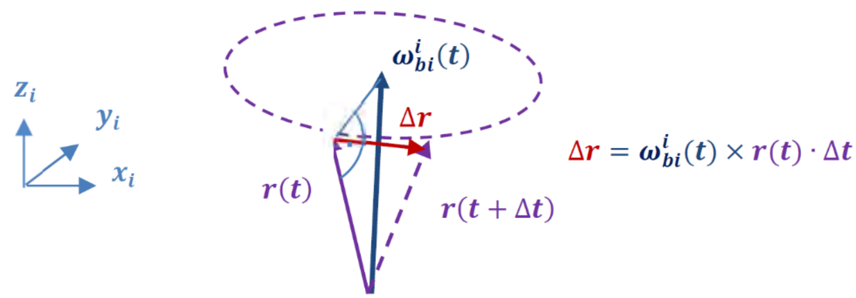

Fig. 2 Change of a vector, $\boldsymbol{\Delta r}$, from $\boldsymbol{r(t)}$ to $\boldsymbol{r(t+\Delta t)}$ due to rotation around rate vector $\boldsymbol{\omega_{bi}^{i}(t)}$.

$$\dot{r} = \tilde{\omega}_{bi}^{i} \cdot r, \tag{1}$$

in which the $\tilde{\omega}_{bi}^{i}$ is the so called "skew-matrix" or "cross product matrix" of vector ${\omega_{bi}^{i}}^{T} = \begin{bmatrix} \omega_{bi\,x}^{i} & \omega_{bi\,y}^{i} & \omega_{bi\,z}^{i} \end{bmatrix}$ and is defined as

$$\tilde{\boldsymbol{\omega}}_{\boldsymbol{bi}}^{\boldsymbol{i}} := \begin{bmatrix} 0 & -\omega_{bi\,z}^{i} & +\omega_{bi\,y}^{i} \\ +\omega_{bi\,z}^{i} & 0 & -\omega_{bi\,x}^{i} \\ -\omega_{bi\,y}^{i} & +\omega_{bi\,x}^{i} & 0 \end{bmatrix}. \tag{2}$$

Note that $\omega_{bi}^{i}(t)$ is the rotation of the body frame wrt inertial frame and expressed in the inertial frame. Now consider not only one vector $r(t)$ but three orthonormal vectors x_b, y_b, z_b spanning the body fixed frame vectors collected as column vectors in the transformation matrix from the body-frame into the inertial frame,

$$T_b^i(t) = \begin{bmatrix} | & | & | \\ x_b(t) & y_b(t) & z_b(t) \\ | & | & | \end{bmatrix} \tag{3}$$

leading from Eq. (1) then to

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$$\dot{T}_b^i(t) = \tilde{\omega}_{bi}^i \cdot T_b^i(t). \tag{4}$$

Eq. (4) is the so called *kinematic differential equation*. There are different versions of the kinematic differential equation: Exchange of indices i, b yields the alternative formulation in the body frame,

$$\dot{T}_i^b(t) = \tilde{\omega}_{ib}^b \cdot T_i^b(t) \tag{5}$$

However, since rates are usually measured w.r.t. the inertial frame the rate

$$\omega_{ib}^b = -\omega_{bi}^b \tag{6}$$

is used. This leads to the version used in [2]:

$$\dot{T}_i^b(t) = -\tilde{\boldsymbol{\omega}}_{\boldsymbol{bi}}^{\boldsymbol{b}} \cdot T_i^b(t) \tag{7}$$

2. The original problem statement

Possible "optimal" slew maneuvers taking into account Eq. (4) are to be derived which satisfy the following "physical" boundary conditions, which are predefined:

a) Attitude boundary conditions

$$T_i^b(t=0) = T_{i\,0}^b \qquad T_i^b(t=t_1) = T_{i\,1}^b \tag{8}$$

b) Rate boundary conditions

$$\omega_{bi}^i(t=0) = \omega_{bi0}^i \qquad \omega_{bi}^i(t=t_1) = \omega_{bi1}^i \tag{9}$$

c) Rate acceleration boundary conditions

$$\dot{\omega}_{bi}^i(t=0) = \dot{\omega}_{bi0}^i \qquad \dot{\omega}_{bi}^i(t=t_1) = \dot{\omega}_{bi1}^i \tag{10}$$

Qualitatively, a slew is "optimal", if the torque is large for small instances of time only.

3. Notation of the kinematic differential equation in terms of row vectors

The transformation matrix $T_b^i(t)$ is now expressed in terms of its row vectors,

$$T_b^i = \begin{bmatrix} - {v_1}^T - \\ - {v_2}^T - \\ - {v_3}^T - \end{bmatrix} \tag{11}$$

and its derivative is then

$$T_b^i = \begin{bmatrix} - {v_1}^T - \\ - {v_2}^T - \\ - {v_3}^T - \end{bmatrix}. \tag{12}$$

Multiplying Eq. (4) on both sides from the right with ${T_b^i}^T$ leads to

$$\dot{T}_b^i \cdot T_i^b = \begin{bmatrix} - {\dot{v}_1}^T - \\ - {\dot{v}_2}^T - \\ - {\dot{v}_3}^T - \end{bmatrix} \cdot [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} {\dot{v}_1}^T \cdot v_1 & {\dot{v}_1}^T \cdot v_2 & {\dot{v}_1}^T \cdot v_3 \\ {\dot{v}_2}^T \cdot v_1 & {\dot{v}_2}^T \cdot v_2 & {\dot{v}_2}^T \cdot v_3 \\ {\dot{v}_3}^T \cdot v_1 & {\dot{v}_3}^T \cdot v_2 & {\dot{v}_3}^T \cdot v_3 \end{bmatrix} = \tilde{\omega} \tag{13}$$

while the indices of the rate vector are from now on omitted.

Since the vectors of a transformation matrix are mutual perpendicular to each other and have unity length, the following equations hold:

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$$v_i^T \cdot v_i = 1 \rightarrow \frac{d(v_i^T \cdot v_i)}{dt} = \dot{v}_i^T \cdot v_i + v_i^T \cdot \dot{v}_i = 2\dot{v}_i^T \cdot v_i = 0 \tag{14}$$

$$v_i^T \cdot v_j = 0 \rightarrow \frac{d\left(v_i^T \cdot v_j\right)}{dt} = \dot{v}_i^T \cdot v_j + v_i^T \cdot \dot{v}_j = 0 \rightarrow \dot{v}_i^T \cdot v_j = -v_i^T \cdot \dot{v}_j. \tag{15}$$

Using Eqs. (13)-(15) and (2) yields

$$\begin{bmatrix} 0 & \dot{v}_1^T \cdot v_2 & \dot{v}_1^T \cdot v_3 \\ \dot{v}_2^T \cdot v_1 & 0 & \dot{v}_2^T \cdot v_3 \\ \dot{v}_3^T \cdot v_1 & \dot{v}_3^T \cdot v_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & +\omega_y \\ +\omega_z & 0 & -\omega_x \\ -\omega_y & +\omega_x & 0 \end{bmatrix}, \tag{16}$$

leading to

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{v}_3^T \cdot v_2 & = & -\dot{v}_2^T \cdot v_3 \\ \dot{v}_1^T \cdot v_3 & = & -\dot{v}_3^T \cdot v_1 \\ \dot{v}_2^T \cdot v_1 & = & -\dot{v}_1^T \cdot v_2 \end{bmatrix} \tag{17}$$

and for the rate acceleration

$$\dot{\omega} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \ddot{v}_3^T \cdot v_2 + \dot{v}_3^T \cdot \dot{v}_2 & = & -\ddot{v}_2^T \cdot v_3 - \dot{v}_2^T \cdot \dot{v}_3 \\ \ddot{v}_1^T \cdot v_3 + \dot{v}_1^T \cdot \dot{v}_3 & = & -\ddot{v}_3^T \cdot v_1 - \dot{v}_3^T \cdot \dot{v}_1 \\ \ddot{v}_2^T \cdot v_1 + \dot{v}_2^T \cdot \dot{v}_1 & = & -\ddot{v}_1^T \cdot v_2 - \dot{v}_1^T \cdot \dot{v}_2 \end{bmatrix}. \tag{18}$$

Reverting the logic of the derivation from above one may come to the conclusion: Any rate vector $\omega(t)$ which can be expressed in its components by the scalar products of the vectors given in Eq. (17) has “automatically” the solution $T_b^i(t)$ given in Eq. (11). This is the basic idea of the parametrization: Suitable vector couples v_1, v_2, v_3 are defined which define at the same time the corresponding rate vector ω: Then, the kinematic differential equation in Eq. (2) is solved *exactly* without the need of numerical integration.

Note that this parametrization can also be performed in the body frame using the corresponding kinematic differential equation in body frame in Eq. (7).

4. Defining suitable vector triples v_1, v_2, v_3-parametrization solving the kinematic differential equation

The idea is to express vector v_1 component wise as polynomials with coefficients to be determined and a polynomial order n to be selected:

$$v_1^T = \frac{\bar{v}_1^T}{|\bar{v}_1|} = \frac{[\bar{v}_{1x} \quad \bar{v}_{1y} \quad \bar{v}_{1z}]}{\sqrt{(\bar{v}_{1x})^2 + \left(\bar{v}_{1y}\right)^2 + (\bar{v}_{1z})^2}} \text{ and for } i = x, y, z \tag{19}$$

$$\bar{v}_{1i} \coloneqq c_{1i\,0} \cdot t^0 + c_{1i\,1} \cdot t^1 + \cdots c_{1i\,n} \cdot t^n = \underbrace{[t^n \quad t^{n-1} \quad \ldots \quad t^2 \quad t^1 \quad t^0]}_{=:h^T} \cdot \underbrace{\begin{bmatrix} c_{1i\,n} \\ c_{1i\,n-1} \\ \vdots \\ c_{1i\,1} \\ c_{1i\,0} \end{bmatrix}}_{=:c_{1i}} = h^T \cdot c_{1i} \tag{20}$$

$$\dot{\bar{v}}_{1i} = \dot{h}^T \cdot c_{1i} = \underbrace{[nt^{n-1} \quad (n-1)t^{n-2} \quad \ldots \quad 2t^1 \quad 1 \quad 0]}_{=:\dot{h}^T} \cdot c_{1i} \tag{21}$$

$$\ddot{\bar{v}}_{1i} = \ddot{h}^T \cdot c_{1i} = \underbrace{[(n-1)nt^{n-2} \quad (n-2)(n-1)t^{n-3} \quad \ldots \quad 2 \quad 0 \quad 0]}_{=:\ddot{h}^T} \cdot c_{1i} \tag{22}$$

Note that vectors v_1 and $\bar{v}_1$ point into the same direction. The time vector h has the dimension $n\ x\ 1$.

Similar vector v_2 is defined to be perpendicular to vector v_1 using the skew matrix of $\bar{v}_1$:

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$$v_2{}^T = \frac{\bar{v}_2{}^T}{\sqrt{(\bar{v}_{2x})^2 + \left(\bar{v}_{2y}\right)^2 + (\bar{v}_{2z})^2}}$$

$$\bar{v}_2 = \alpha \cdot \begin{bmatrix} 0 \\ +\bar{v}_{1z} \\ -\bar{v}_{1y} \end{bmatrix} + \beta \cdot \begin{bmatrix} +\bar{v}_{1z} \\ 0 \\ -\bar{v}_{1x} \end{bmatrix} + \gamma \cdot \begin{bmatrix} +\bar{v}_{1y} \\ -\bar{v}_{1x} \\ 0 \end{bmatrix} = \tilde{\bar{v}}_1 \cdot \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}}_{=:\delta} \tag{23}$$

Clearly, the scalar product $\bar{v}_1{}^T \cdot \bar{v}_2 = \bar{v}_1{}^T \cdot \tilde{\bar{v}}_1 \cdot \delta = 0$ for any vector δ. The values α, β, γ are expressed again by time dependent polynomials with coefficients $c_\alpha, c_\beta, c_\gamma$ to be determined:

$$\alpha(t) := h(t)^T \cdot c_\alpha \qquad \beta(t) := h(t)^T \cdot c_\beta \qquad \gamma(t) := h(t)^T \cdot c_\gamma \tag{24}$$

The third vector v_3 does not contain any degree of freedom and simply follows the right-hand-rule:

$$v_3 = \tilde{v}_1 \cdot v_2 \tag{25}$$

$$\dot{v}_3 = \dot{\tilde{v}}_1 \cdot v_2 + \tilde{v}_1 \cdot \dot{v}_2 \tag{26}$$

$$\ddot{v}_3 = \ddot{\tilde{v}}_1 \cdot v_2 + \dot{\tilde{v}}_1 \cdot \dot{v}_2 + \dot{\tilde{v}}_1 \cdot \dot{v}_2 + \tilde{v}_1 \cdot \ddot{v}_2 = \ddot{\tilde{v}}_1 \cdot v_2 + 2\dot{\tilde{v}}_1 \cdot \dot{v}_2 + \tilde{v}_1 \cdot \ddot{v}_2 \tag{27}$$

Now, the physical boundary conditions on attitude, rate and rate acceleration need to be taken into account from the vector couples v_1, v_2, in which the coefficients c_{1x}, c_{1y} c_{1z} and c_α, c_β, c_γ, respectively, need to be determined.

C. Mapping of physical boundary conditions into vector boundary conditions

5. Mapping of <u>attitude</u> boundary conditions into $\bar{v}_1(0), \bar{v}_1(t_1)$ and $\delta(0), \delta(t_1)$

The attitude at slew start and slew end is predefined:

$$T_b^i(t=0) = T_{b\,0}^i = \begin{bmatrix} - & v_{10}{}^T & - \\ - & v_{20}{}^T & - \\ - & v_{30}{}^T & - \end{bmatrix} \tag{28}$$

$$T_b^i(t=t_1) = T_{b\,1}^i = \begin{bmatrix} - & v_{11}{}^T & - \\ - & v_{21}{}^T & - \\ - & v_{31}{}^T & - \end{bmatrix} \tag{29}$$

Thus, the initial and the final value – index 0 and index 1, respectively, of vector v_1 is fixed:

$$\bar{v}_{10}{}^T = v_{10}{}^T = T_{b\,0}^i(1,:) \tag{30}$$

$$\bar{v}_{11}{}^T = v_{11}{}^T = T_{b\,1}^i(1,:). \tag{31}$$

The same holds for vector v_2, but for this vector the boundary values of δ need to be determined:

$$\bar{v}_2(t=0) = v_{20} = \tilde{\bar{v}}_{10} \cdot \delta_0 = \tilde{v}_{10}\delta_0 \tag{32}$$

Since v_{20} is perpendicular to v_{10}, one particular solution vector δ_{0p} is $\delta_{0p} = -v_{30}$ from the right hand rule. Since matrix $\tilde{v}_{10}$ has a loss of rank from the zeros in the main diagonal, vector δ_0 cannot uniquely be determined. There is one degree of freedom in it: An arbitrary contribution χ_{13} from v_{10}, the so called null-space matrix (here: null space vector) of matrix $\tilde{v}_{10}$, can be added to vector δ_0 without violating Eq. (32):

$$v_{20} = \tilde{v}_{10} \cdot \delta_0 = \tilde{v}_{10} \cdot \left(\delta_{0p} + \chi_1 \cdot v_{10}\right) = \tilde{v}_{10}(-v_{30} + \chi_{13} \cdot v_{10}) \tag{33}$$

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Thus, for the initial and the final values of coefficient vector $\delta(t)$ the following boundary conditions hold:

$$\delta_0 = \delta_{0p} + \chi_{13} \cdot v_{10} \text{ with } \delta_{0p} = -v_{30}, \tag{34}$$

$$\delta_1 = \delta_{1p} + \chi_{14} \cdot v_{10} \text{ with } \delta_{1p} = -v_{31}. \tag{35}$$

6. Mapping of rate boundary conditions into $\dot{\bar{v}}_1(0), \dot{\bar{v}}_1(t_1)$ and $\dot{\delta}(0), \dot{\delta}(t_1)$

In the first glance, from Eq. (17) the connection can be derived for the initial condition in y, z- component of rate ω and vector $\dot{v}_1$ but not $\dot{\bar{v}}_1$:

$$\begin{bmatrix} \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} +{v_3}^T \cdot \dot{v}_1 \\ -{v_2}^T \cdot \dot{v}_1 \end{bmatrix} = \begin{bmatrix} +{v_3}^T \\ -{v_2}^T \end{bmatrix} \cdot \dot{v}_1 \tag{36}$$

From Eq. (19) the derivative $\dot{v}_1$ can computed in general at start using with $|\bar{v}_1| = 1$:

$$v_1 = \frac{\bar{v}_1}{|\bar{v}_1|} \tag{37}$$

$$\dot{v}_1 = \left(\dot{\bar{v}}_1 \cdot |\bar{v}_1| - \bar{v}_1 \cdot |\dot{\bar{v}}_1|\right) \Big/ |\bar{v}_1|^2 \tag{38}$$

$$|\dot{\bar{v}}_1| := \frac{d|\bar{v}_1|}{dt} = \frac{d}{dt}\sqrt{{\bar{v}_1}^T \bar{v}_1} = \frac{1}{2\sqrt{{\bar{v}_1}^T \bar{v}_1}} \cdot 2{\bar{v}_1}^T \dot{\bar{v}}_1 \frac{{\bar{v}_1}^T \dot{\bar{v}}_1}{\sqrt{{\bar{v}_1}^T \bar{v}_1}} \cdot \tag{39}$$

$$\dot{v}_{10} = \dot{\bar{v}}_{10} - \bar{v}_{10} \cdot {\bar{v}_{10}}^T \dot{\bar{v}}_{10} = \left(E - \bar{v}_{10} \cdot {\bar{v}_{10}}^T\right) \cdot \dot{\bar{v}}_{10} \qquad E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{40}$$

Note that vectors $\dot{v}_1$ and $\dot{\bar{v}}_1$ do *not* point in general into the same direction.

Inserting Eq. (38) into Eq. (36) and reminding that ${v_3}^T \cdot \bar{v}_1 = {v_3}^T \cdot \bar{v}_1 = 0$ gives

$$\begin{bmatrix} \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} +{v_3}^T \\ -{v_2}^T \end{bmatrix} \cdot \left(\dot{\bar{v}}_1 \cdot |\bar{v}_1| - \bar{v}_1 \cdot |\dot{\bar{v}}_1|\right) \Big/ |\bar{v}_1|^2 = \begin{bmatrix} +{v_3}^T \\ -{v_2}^T \end{bmatrix} \cdot \left(\dot{\bar{v}}_1\right) \Big/ |\bar{v}_1| \cdot \tag{41}$$

At $t = 0$ and $t = t_1$ the values of $\bar{v}_1$ are selected to be one, otherwise the initial and final attitude condition could not be satisfied. Then, Eq. (41) turns into

$$\begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix} = \begin{bmatrix} +{v_{30}}^T \\ -{v_{20}}^T \end{bmatrix} \cdot \dot{\bar{v}}_{10} \tag{42}$$

From Eq. (42), the unknown initial vector $\dot{\bar{v}}_{10}$ can be determined. A particular solution for $\dot{\bar{v}}_{10}$, $\dot{\bar{v}}_{10\,p}$, is

$$\dot{\bar{v}}_{10\,p} = \begin{bmatrix} +v_{30} & -v_{20} \end{bmatrix} \begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix} \tag{43}$$

The *general* solution including the null space vector is then at start and end

$$\dot{\bar{v}}_{10} = \dot{\bar{v}}_{10\,p} + \chi_1 \cdot v_{10} \tag{44}$$

$$\dot{\bar{v}}_{11} = \dot{\bar{v}}_{11\,p} + \chi_4 \cdot v_{11} \tag{45}$$

with

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$$\dot{\bar{v}}_{11\,p} = [+v_{31} \quad -v_{21}] \begin{bmatrix} \omega_{y1} \\ \omega_{z1} \end{bmatrix} \tag{46}$$

To cover the initial x-rate component vector $\dot{v}_2$ in Eq. (17) needs to be determined:

$$\omega_x = -{v_3}^T \cdot \dot{v}_2 \tag{47}$$

The derivative $\dot{v}_2$ can be determined as in Eq. (38)

$$\dot{v_2} = {(\dot{\bar{v}}_2 \cdot |\bar{v}_2| - \bar{v}_2 \cdot |\dot{\bar{v}}_2|)}\Big/{|\bar{v}_2|^2} \tag{48}$$

and inserting in Eq. (47) gives

$$\omega_x = -{v_3}^T \cdot {(\dot{\bar{v}}_2)}\Big/{|\bar{v}_2|} \tag{49}$$

Vector $\dot{\bar{v}}_2$ is computed from Eq. (23):

$$\bar{v}_2 = \tilde{\bar{v}}_1 \cdot \delta \tag{50}$$

$$\dot{\bar{v}}_2 = \dot{\tilde{\bar{v}}}_1 \cdot \delta + \tilde{\bar{v}}_1 \cdot \dot{\delta} \tag{51}$$

and for completeness

$$\ddot{\bar{v}}_2 = \ddot{\tilde{\bar{v}}}_1 \cdot \delta + \dot{\tilde{\bar{v}}}_1 \cdot \dot{\delta} + \dot{\tilde{\bar{v}}}_1 \cdot \dot{\delta} + \tilde{\bar{v}}_1 \cdot \ddot{\delta} = \ddot{\tilde{\bar{v}}}_1 \cdot \delta + 2\dot{\tilde{\bar{v}}}_1 \cdot \dot{\delta} + \tilde{\bar{v}}_1 \cdot \ddot{\delta}. \tag{52}$$

At the boundaries Eq. (49) turns with Eq. (51) into

$$\omega_{x0} = -{v_{30}}^T \cdot \dot{\bar{v}}_{20} = -{v_{30}}^T (\dot{\tilde{\bar{v}}}_{10} \cdot \delta_0 + \tilde{\bar{v}}_{10} \cdot \dot{\delta}_0 \cdot \dot{\delta}_0) \tag{53}$$

Rearranging for the only unknown variable $\dot{\delta}_0$ yields

$$\omega_{x0} + {v_{30}}^T \dot{\tilde{\bar{v}}}_{10} \delta_0 = -{v_{30}}^T \tilde{\bar{v}}_{10} \cdot \dot{\delta}_0 \tag{54}$$

From Eq. (54) the unknown initial vector $\dot{\delta}_0$ can be determined. Because $-{v_{30}}^T \tilde{\bar{v}}_{10} = -{v_{20}}^T$ a particular solution for $\dot{\delta}_0$, $\dot{\delta}_{0\,p1}$, is gained by using Eqs. (34), (43) and (44); additionally, all terms involving scalar products with vectors perpendicular to each other are removed:

$$\begin{aligned}
\dot{\delta}_{0\,p1} &= -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T \dot{\tilde{\bar{v}}}_{10}\, \delta_0\right) = -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T \left(\dot{\bar{v}}_{10p} \widetilde{+ \chi_1} \cdot v_{10}\right) (-v_{30} + \chi_{13} \cdot v_{10})\right) \\
\dot{\delta}_{0\,p1} &= -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T \left(\widetilde{[+v_{30} \quad -v_{20}] \begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix}} + \chi_1 \cdot v_{10}\right) (-v_{30} + \chi_{13} \cdot v_{10})\right) \\
\dot{\delta}_{0\,p1} &= -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T (\omega_{y0} \cdot \tilde{v}_{30} - \omega_{z0} \cdot \tilde{v}_{20} + \chi_1 \cdot \tilde{v}_{10})(-v_{30} + \chi_{13} \cdot v_{10})\right) \\
\dot{\delta}_{0\,p1} &= -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T (-\omega_{z0} \cdot \tilde{v}_{20} + \chi_1 \cdot \tilde{v}_{10})(+\chi_{13} \cdot v_{10})\right) = -v_{20} \cdot \left(\omega_{x0} + {v_{30}}^T (\omega_{z0} \chi_{13} \cdot v_{30})\right) \\
\dot{\delta}_{0\,p1} &= -v_{20} \cdot (\omega_{x0} + \omega_{z0} \cdot \chi_{13})
\end{aligned} \tag{55}$$

Thus, this particular solution $\dot{\delta}_{0\,p1}$ also depends on the attitude boundary condition involving χ_{13}.

The null space matrix of $-{v_{30}}^T \tilde{\bar{v}}_{10}$ in Eq. (55) consists of the vectors v_{10} and v_{30}.For the complete and general solution the particular solution $\dot{\delta}_{0\,p1}$is split in $\dot{\delta}_{0\,p} = -\omega_{x0} \cdot v_{20}$ and the part with χ_{13} , $-v_{20} \cdot \omega_{z0} \cdot \chi_{13}$, is included in matrix D_{10} and D_{10}, respectively, leading to

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$$\dot{\delta}_0 = \dot{\delta}_{0p} + D_{10} \cdot \begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_{13} \end{bmatrix} \qquad \dot{\delta}_{0p} = -\omega_{x0} \cdot v_{20} \qquad D_{10} = [v_{10} \quad v_{30} \quad -\omega_{z0} \cdot v_{20}] \tag{56}$$

$$\dot{\delta}_1 = \dot{\delta}_{1p} + D_{11} \cdot \begin{bmatrix} \chi_5 \\ \chi_6 \\ \chi_{14} \end{bmatrix} \qquad \dot{\delta}_{1p} = -\omega_{x1} \cdot v_{21} \qquad D_{11} = [v_{11} \quad v_{31} \quad -\omega_{z1} \cdot v_{21}] \tag{57}$$

7. *Mapping of* <u>*rate acceleration*</u> *boundary conditions into* $\dddot{\bar{v}}_1(0), \dddot{\bar{v}}_1(t_1)$ *and* $\ddot{\delta}(0), \ddot{\delta}(t_1)$
The derivative of the rate signal $\omega(t)$ in Eq. (18) forms the connection from the physical boundary conditions:

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{v}_3{}^T \cdot v_2 & = & -\dot{v}_2{}^T \cdot v_3 \\ \dot{v}_1{}^T \cdot v_3 & = & -\dot{v}_3{}^T \cdot v_1 \\ \dot{v}_2{}^T \cdot v_1 & = & -\dot{v}_1{}^T \cdot v_2 \end{bmatrix}, \tag{58}$$

$$\dot{\omega} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \ddot{v}_3{}^T \cdot v_2 + \dot{v}_3{}^T \cdot \dot{v}_2 & = & -\ddot{v}_2{}^T \cdot v_3 - \dot{v}_2{}^T \cdot \dot{v}_3 \\ \ddot{v}_1{}^T \cdot v_3 + \dot{v}_1{}^T \cdot \dot{v}_3 & = & -\ddot{v}_3{}^T \cdot v_1 - \dot{v}_3{}^T \cdot \dot{v}_1 \\ \ddot{v}_2{}^T \cdot v_1 + \dot{v}_2{}^T \cdot \dot{v}_1 & = & -\ddot{v}_1{}^T \cdot v_2 - \dot{v}_1{}^T \cdot \dot{v}_2 \end{bmatrix}. \tag{59}$$

Since v_3 involves more expressions from Eq. (59) the following rate acceleration component-combination is used:

$$\dot{\omega} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} -v_3{}^T \cdot \ddot{v}_2 - \dot{v}_2{}^T \cdot \dot{v}_3 \\ +v_3{}^T \cdot \ddot{v}_1 + \dot{v}_1{}^T \cdot \dot{v}_3 \\ -v_2{}^T \cdot \ddot{v}_1 - \dot{v}_1{}^T \cdot \dot{v}_2 \end{bmatrix} \tag{60}$$

The initial rate acceleration conditions are

$$\begin{bmatrix} \dot{\omega}_{x0} \\ \dot{\omega}_{y0} \\ \dot{\omega}_{z0} \end{bmatrix} = \begin{bmatrix} -v_{30}{}^T \cdot \ddot{v}_{20} - \dot{v}_{20}{}^T \cdot \dot{v}_{30} \\ +v_{30}{}^T \cdot \ddot{v}_{10} + \dot{v}_{10}{}^T \cdot \dot{v}_{30} \\ -v_{20}{}^T \cdot \ddot{v}_{10} - \dot{v}_{10}{}^T \cdot \dot{v}_{20} \end{bmatrix}. \tag{61}$$

Solving Eq. (61) for vector $\ddot{v}_{10}$ yields

$$\begin{bmatrix} \dot{\omega}_{y0} - \dot{v}_{10}{}^T \cdot \dot{v}_{30} \\ \dot{\omega}_{z0} + \dot{v}_{10}{}^T \cdot \dot{v}_{20} \end{bmatrix} = \begin{bmatrix} +v_{30}{}^T \\ -v_{20}{}^T \end{bmatrix} \ddot{v}_{10}. \tag{62}$$

In order to get an expression for $\dddot{\bar{v}}_{10}$ instead of $\ddot{v}_{10}$ the derivatives $\ddot{v}_1$ are computed from Eq. (38):

$$\dot{v}_1 = \frac{1}{|\bar{v}_1|^2} \cdot \left(\dot{\bar{v}}_1 \cdot |\bar{v}_1| - \bar{v}_1 \cdot |\dot{\bar{v}}_1|\right) \tag{63}$$

$$\ddot{v}_1 = \frac{1}{|\bar{v}_1|^4} \cdot \left(\left(\ddot{\bar{v}}_1 \cdot |\bar{v}_1| + \dot{\bar{v}}_1 \cdot |\dot{\bar{v}}_1| - \dot{\bar{v}}_1 \cdot |\dot{\bar{v}}_1| - \bar{v}_1 \cdot |\ddot{\bar{v}}_1|\right)|\bar{v}_1|^2 - (\dot{\bar{v}}_1 \cdot |\bar{v}_1| - \bar{v}_1 \cdot |\dot{\bar{v}}_1|)2|\bar{v}_1||\dot{\bar{v}}_1|\right)$$

$$\ddot{v}_1 = \frac{1}{|\bar{v}_1|^4} \cdot \left(\ddot{\bar{v}}_1 \cdot |\bar{v}_1|^3 - \dot{\bar{v}}_1 \cdot 2|\bar{v}_1|^2|\dot{\bar{v}}_1| + \bar{v}_1 \cdot (2|\dot{\bar{v}}_1|^2|\bar{v}_1| - |\ddot{\bar{v}}_1||\bar{v}_1|^2\right) \tag{64}$$

and using Eq. (39)

$$\ddot{v}_1 = \frac{1}{|\bar{v}_1|^4} \cdot \left(\ddot{\bar{v}}_1 \cdot |\bar{v}_1|^3 - \dot{\bar{v}}_1 \cdot 2|\bar{v}_1|^2 \frac{\bar{v}_1{}^T \dot{\bar{v}}_1}{\sqrt{\bar{v}_1{}^T \bar{v}_1}} + \bar{v}_1 \cdot (2 \frac{\left(\bar{v}_1{}^T \dot{\bar{v}}_1\right)^2}{\sqrt{\bar{v}_1{}^T \bar{v}_1}} - |\ddot{\bar{v}}_1||\bar{v}_1|^2)\right) \tag{65}$$

with its initial values

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$$\ddot{v}_{10} = \left(\ddot{\bar{v}}_{10} - \dot{\bar{v}}_{10} \cdot 2\bar{v}_{10}{}^T \dot{\bar{v}}_{10} + \bar{v}_{10} \cdot (2\left(\bar{v}_{10}{}^T \dot{\bar{v}}_{10}\right)^2 - |\ddot{\bar{v}}_{10}|)\right). \tag{66}$$

Inserting Eq. (66) in Eq. (62) gives

$$\begin{aligned} \begin{bmatrix} \dot{\omega}_{y0} - \dot{v}_{10}{}^T \cdot \dot{v}_{30} \\ \dot{\omega}_{z0} + \dot{v}_{10}{}^T \cdot \dot{v}_{20} \end{bmatrix} &= \begin{bmatrix} +v_{30}{}^T \\ -v_{20}{}^T \end{bmatrix} \left(\ddot{\bar{v}}_{10} - \dot{\bar{v}}_{10} \cdot 2\bar{v}_{10}{}^T \dot{\bar{v}}_{10} + \bar{v}_{10} \cdot (2\left(\bar{v}_{10}{}^T \dot{\bar{v}}_{10}\right)^2 - |\ddot{\bar{v}}_{10}|)\right) \\ &= \begin{bmatrix} +v_{30}{}^T \\ -v_{20}{}^T \end{bmatrix} \left(\ddot{\bar{v}}_{10} - \dot{\bar{v}}_{10} \cdot 2\bar{v}_{10}{}^T \dot{\bar{v}}_{10}\right) \end{aligned} \tag{67}$$

rearranging terms finally

$$\begin{bmatrix} \dot{\omega}_{y0} \underbrace{-\dot{v}_{30}{}^T \cdot \dot{v}_{10}}_{A1} \underbrace{+2v_{30}{}^T \dot{\bar{v}}_{10} \cdot \bar{v}_{10}{}^T \dot{\bar{v}}_{10}}_{A2} \\ \dot{\omega}_{z0} \underbrace{+\dot{v}_{10}{}^T \cdot \dot{v}_{20}}_{A3} \underbrace{-2v_{20}{}^T \dot{\bar{v}}_{10} \cdot \bar{v}_{10}{}^T \dot{\bar{v}}_{10}}_{A4} \end{bmatrix} = \begin{bmatrix} +v_{30}{}^T \\ -v_{20}{}^T \end{bmatrix} \ddot{\bar{v}}_{10}. \tag{68}$$

Expressions $A1, A2, A3, A4$ are now solved for the unknown χ-values in Eqs. (44), (45), (56), (57).

$A1=-\dot{v}_{30}{}^T \cdot \dot{v}_{10}$:

Inserting Eq. (26) at *t=0*

$$\begin{aligned} A1 &= -\dot{v}_{30}{}^T \cdot \dot{v}_{10} \\ &= -\left(\dot{\tilde{v}}_{10} \cdot v_{20} + \tilde{v}_{10} \cdot \dot{v}_{20}\right)^T \cdot \dot{v}_{10} = -(\tilde{v}_{10} \cdot \dot{v}_{20})^T \cdot \dot{v}_{10} = \dot{v}_{20}{}^T \tilde{v}_{10} \dot{v}_{10} \\ &\underbrace{=}_{eqs.\ (40),(44)} \dot{v}_{20}{}^T \tilde{v}_{10}\left(E - \bar{v}_{10} \cdot \bar{v}_{10}{}^T\right) \cdot \dot{\bar{v}}_{10} = \dot{v}_{20}{}^T \tilde{v}_{10}\left(E - \bar{v}_{10} \cdot \bar{v}_{10}{}^T\right)\left(\begin{bmatrix} +v_{30} & -v_{20} \end{bmatrix} \begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix} + \chi_1 \cdot v_{10}\right) \\ &= \dot{v}_{20}{}^T \tilde{v}_{10} \left(\begin{bmatrix} +v_{30} & -v_{20} \end{bmatrix} \begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix}\right) = \dot{v}_{20}{}^T \left(\begin{bmatrix} -v_{20} & -v_{30} \end{bmatrix} \begin{bmatrix} \omega_{y0} \\ \omega_{z0} \end{bmatrix}\right) \\ &\underbrace{=}_{eq.(40)\ for\ v_2} \begin{bmatrix} \omega_{y0} & \omega_{z0} \end{bmatrix} \begin{bmatrix} -v_{20}{}^T \\ -v_{30}{}^T \end{bmatrix} \dot{v}_{20} = \begin{bmatrix} \omega_{y0} & \omega_{z0} \end{bmatrix} \begin{bmatrix} -v_{20}{}^T \\ -v_{30}{}^T \end{bmatrix} \left(E - \bar{v}_{20} \cdot \bar{v}_{20}{}^T\right) \cdot \dot{\bar{v}}_{20} \\ &= -\omega_{z0} \cdot v_{30}{}^T \cdot \dot{\bar{v}}_{20} \\ &\underbrace{=}_{eqs.(40),(51)\ at\ t=0} -\omega_{z0} \cdot v_{30}{}^T \cdot \left(\dot{\tilde{\bar{v}}}_{10} \cdot \delta_0 + \tilde{\bar{v}}_{10} \cdot \dot{\delta}_0\right) \\ &\underbrace{=}_{eqs.(34)\ (44)} -\omega_{z0} \cdot v_{30}{}^T \cdot \left(\left(+\omega_{y0}\tilde{v}_{30} - \omega_{z0}\tilde{v}_{20} + \chi_1 \cdot \tilde{v}_{10}\right) \cdot (-v_{30} + \chi_{13} \cdot v_{10}) + \tilde{\bar{v}}_{10} \cdot \dot{\delta}_0\right) \\ &\underbrace{=}_{eq.(56)} -\omega_{z0} \cdot \left((+\omega_{z0} v_{10}{}^T + \chi_1 \cdot v_{20}{}^T) \cdot (-v_{30} + \chi_{13} \cdot v_{10}) + v_{30}{}^T \cdot \tilde{\bar{v}}_{10} \cdot \dot{\delta}_0\right) \\ &= -\omega_{z0} \cdot \left(+\omega_{z0} \cdot \chi_{13} + v_{30}{}^T \cdot \tilde{\bar{v}}_{10} \cdot \left(-\omega_{x0} \cdot v_{20} + \begin{bmatrix} v_{10} & v_{30} & -\omega_{z0} \cdot v_{20} \end{bmatrix} \cdot \begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_{13} \end{bmatrix}\right)\right) \\ &= -\omega_{z0} \cdot \left(+\omega_{z0} \cdot \chi_{13} + v_{20}{}^T \left(-\omega_{x0} \cdot v_{20} + \begin{bmatrix} v_{10} & v_{30} & -\omega_{z0} \cdot v_{20} \end{bmatrix} \cdot \begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_{13} \end{bmatrix}\right)\right) \\ &= -\omega_{z0} \cdot (+\omega_{z0} \cdot \chi_{13} - \omega_{x0} - \omega_{z0} \cdot \chi_{13}) \\ &= +\omega_{x0} \cdot \omega_{z0} \end{aligned}$$

$$A1 = +\omega_{x0} \cdot \omega_{z0} \tag{69}$$

In a similar way the following simple expressions are received:

$$A2 = 2\omega_{y0} \cdot \chi_1 \tag{70}$$

$$A3 = -\omega_{x0} \cdot \omega_{y0} \tag{71}$$

$$A4 = +2\omega_{z0} \cdot \chi_1 \tag{72}$$

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Inserting the last four eqs. into Eq. (68) gives

$$\begin{bmatrix}\dot{\omega}_{y0}+\omega_{x0}\cdot\omega_{z0}\\ \dot{\omega}_{z0}-\omega_{x0}\cdot\omega_{y0}\end{bmatrix}+\begin{bmatrix}2\omega_{y0}\\ 2\omega_{z0}\end{bmatrix}\cdot\chi_1=\begin{bmatrix}+{v_{30}}^T\\ -{v_{20}}^T\end{bmatrix}\ddot{\bar{v}}_{10}. \tag{73}$$

For Eq.(73) a particular solution for $\ddot{\bar{v}}_{10}$, $\ddot{\bar{v}}_{10\,\mathrm{p1}}$, is obviously

$$\ddot{\bar{v}}_{10\,\mathrm{p1}}=\begin{bmatrix}+v_{30} & -v_{20}\end{bmatrix}\left(\begin{bmatrix}\dot{\omega}_{y0}+\omega_{x0}\cdot\omega_{z0}\\ \dot{\omega}_{z0}-\omega_{x0}\cdot\omega_{y0}\end{bmatrix}+\begin{bmatrix}2\omega_{y0}\\ 2\omega_{z0}\end{bmatrix}\cdot\chi_1\right) \tag{74}$$

The general solution including null space vector v_{10} scalable with an arbitrary number χ_7 is then

$$\ddot{\bar{v}}_{10}=\ddot{\bar{v}}_{10\,\mathrm{p1}}+\chi_7\cdot v_{10}. \tag{75}$$

Eq. (75) is now separated in terms with and without χ-values, and similar for the final values $\ddot{\bar{v}}_{11}$:

$$\ddot{\bar{v}}_{10}=\ddot{\bar{v}}_{10\mathrm{p}}+U_{20}\begin{bmatrix}\chi_1\\ \chi_7\end{bmatrix};\ddot{\bar{v}}_{10\mathrm{p}}:=\begin{bmatrix}+v_{30} & -v_{20}\end{bmatrix}\begin{bmatrix}\dot{\omega}_{y0}+\omega_{x0}\cdot\omega_{z0}\\ \dot{\omega}_{z0}-\omega_{x0}\cdot\omega_{y0}\end{bmatrix};U_{20}:=\begin{bmatrix}2(\omega_{y0}v_{30}-\omega_{z0}v_{20}) & v_{10}\end{bmatrix} \tag{76}$$

$$\ddot{\bar{v}}_{11}=\ddot{\bar{v}}_{11\mathrm{p}}+U_{21}\begin{bmatrix}\chi_4\\ \chi_{10}\end{bmatrix};\ddot{\bar{v}}_{11\mathrm{p}}:=\begin{bmatrix}+v_{31} & -v_{21}\end{bmatrix}\begin{bmatrix}\dot{\omega}_{y1}+\omega_{x1}\cdot\omega_{z1}\\ \dot{\omega}_{z1}-\omega_{x1}\cdot\omega_{y1}\end{bmatrix};U_{21}:=\begin{bmatrix}2(\omega_{y1}v_{31}-\omega_{z1}v_{21}) & v_{11}\end{bmatrix} \tag{77}$$

Finally, in order to map the x-rate acceleration initial condition $\dot{\omega}_{x0}$ solving Eq. (61) for vector $\ddot{v}_{20}$ yields

$$\dot{\omega}_{x0}+{\dot{v}_{20}}^T\cdot\dot{v}_{30}=-{v_{30}}^T\ddot{v}_{20} \tag{78}$$

Similar to Eq. (66), vector $\ddot{v}_{20}$ is computed to be

$$\ddot{v}_{20}=\left(\ddot{\bar{v}}_{20}-\dot{\bar{v}}_{20}\cdot 2{\bar{v}_{20}}^T\dot{\bar{v}}_{20}+\bar{v}_{20}\cdot(2\left({\bar{v}_{20}}^T\dot{\bar{v}}_{20}\right)^2-|\ddot{\bar{v}}_{20}|)\right) \tag{79}$$

Vector $\ddot{\bar{v}}_{20}$ can be be computed by forming the second derivative in Eq. (56)

$$\ddot{\bar{v}}_{20}=\ddot{\tilde{\bar{v}}}_{10}\cdot\delta_0+\dot{\tilde{\bar{v}}}_{10}\cdot\dot{\delta}_0+\dot{\tilde{\bar{v}}}_{10}\cdot\dot{\delta}_0+\tilde{\bar{v}}_{10}\cdot\ddot{\delta}_0=\ddot{\tilde{\bar{v}}}_{10}\cdot\delta_0+2\dot{\tilde{\bar{v}}}_{10}\cdot\dot{\delta}_0+\tilde{\bar{v}}_{10}\cdot\ddot{\delta}_0 \tag{80}$$

Inserting Eq. (79) in Eq. (78) gives

$$\begin{aligned}\dot{\omega}_{x0}+{\dot{v}_{20}}^T\cdot\dot{v}_{30}&=-{v_{30}}^T\left(\ddot{\bar{v}}_{20}-\dot{\bar{v}}_{20}\cdot 2{\bar{v}_{20}}^T\dot{\bar{v}}_{20}+\bar{v}_{20}\cdot\left(2\left({\bar{v}_{20}}^T\dot{\bar{v}}_{20}\right)^2-|\ddot{\bar{v}}_{20}|\right)\right)\\ &=-{v_{30}}^T{v_{30}}^T\left(\ddot{\bar{v}}_{20}-\dot{\bar{v}}_{20}\cdot 2{\bar{v}_{20}}^T\dot{\bar{v}}_{20}\right)\end{aligned} \tag{81}$$

rearranging terms using Eq. (80) finally

$$\dot{\omega}_{x0}+{\dot{v}_{20}}^T\cdot\dot{v}_{30}=-{v_{30}}^T\left(\ddot{\tilde{\bar{v}}}_{10}\cdot\delta_0+2\dot{\tilde{\bar{v}}}_{10}\cdot\dot{\delta}_0+\tilde{\bar{v}}_{10}\cdot\ddot{\delta}_0-\dot{\bar{v}}_{20}\cdot 2{\bar{v}_{20}}^T\dot{\bar{v}}_{20}\right) \tag{82}$$

$$\dot{\omega}_{x0}+2{v_{30}}^T\dot{\tilde{\bar{v}}}_{10}\dot{\delta}_0+{v_{30}}^T\ddot{\tilde{\bar{v}}}_{10}\delta_0+{\dot{v}_{30}}^T\dot{v}_{20}-2{v_{30}}^T\dot{\bar{v}}_{20}{\bar{v}_{20}}^T\dot{\bar{v}}_{20}=-{v_{20}}^T\tilde{\bar{v}}_{10}\cdot\ddot{\delta}_0=-{v_{20}}^T\cdot\ddot{\delta}_0 \tag{83}$$

while δ_0, $\dot{\delta}_0$, $\tilde{\bar{v}}_{10}$, $\dot{\tilde{\bar{v}}}_{10}$, $\ddot{\tilde{\bar{v}}}_{10}$ are determined in Eqs. (34), (56), (30), (44), (76), and $\ddot{\delta}_0$ is to be determined. Similar to Eqs. (76), (77) the following results can be derived:

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$$\ddot{\delta}_0 = \ddot{\delta}_{0p} + D_{20} \cdot \underbrace{\begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_8 \\ \chi_9 \\ \chi_{13} \end{bmatrix}}_{=:\chi_{\delta Q}}; \qquad \ddot{\delta}_{0p} := v_{20} \cdot \left(-\dot{\omega}_{x0} + \omega_{y0}\omega_{z0}\right);$$
$$D_{20} := \begin{bmatrix} -2\omega_{z0} \cdot v_{20} & 2\omega_{x0} \cdot v_{20} & v_{10} & v_{30} & -\left(\dot{\omega}_{z0} + \omega_{x0} \cdot \omega_{y0}\right) \cdot v_{20} \end{bmatrix} \tag{84}$$

$$\ddot{\delta}_1 = \ddot{\delta}_{1p} + D_{21} \cdot \underbrace{\begin{bmatrix} \chi_5 \\ \chi_6 \\ \chi_{11} \\ \chi_{12} \\ \chi_{14} \end{bmatrix}}_{=:\chi_{\delta R}}; \qquad \ddot{\delta}_{1p} := v_{21} \cdot \left(-\dot{\omega}_{x1} + \omega_{y1}\omega_{z1}\right);$$
$$D_{21} := \begin{bmatrix} -2\omega_{z1} \cdot v_{21} & 2\omega_{x1} \cdot v_{21} & v_{11} & v_{31} & -\left(\dot{\omega}_{z1} + \omega_{x1} \cdot \omega_{y1}\right) \cdot v_{21} \end{bmatrix} \tag{85}$$

Note that again as in Eq. (77), (78) all boundary conditions depend *linearly* on the free and unconstraint design parameters χ_i.

D. Satisfaction of vector boundary conditions

8. Satisfaction of boundary conditions of vector v_1

To summarize, the boundary conditions of the involved polynomials are listed.

For the three polynomials $\bar{v}_{1i}(t) = h^T(t) \cdot c_{1i}$ ($i = x, y, z$ from Eq. (2)) the following vector boundary conditions according to Eqs. (30), (31), (44), (45), (76), (77) hold and restrict the choice of the unknown coefficients c_{1i}:

$$\bar{v}_{10i}(\chi) = h^T(t=0) \cdot c_{1i} \tag{86}$$
$$\bar{v}_{11i}(\chi) = h^T(t=t_1) \cdot c_{1i} \tag{87}$$
$$\dot{\bar{v}}_{10\,i}(\chi) = \dot{h}^T(t=0) \cdot c_{1i} \tag{88}$$
$$\dot{\bar{v}}_{11\,i}(\chi) = \dot{h}^T(t=t_1) \cdot c_{1i} \tag{89}$$
$$\ddot{\bar{v}}_{10\,i}(\chi) = \ddot{h}^T(t=0) \cdot c_{1i} \tag{90}$$
$$\ddot{\bar{v}}_{11\,i}(\chi) = \ddot{h}^T(t=t_1) \cdot c_{1i} \tag{91}$$

Eqs. (95)-(100)can be summarized in matrix notation for each axis $i = x, y, z$

$$\underbrace{\begin{bmatrix} \bar{v}_{10i} \\ \bar{v}_{11i} \\ \dot{\bar{v}}_{10\,i} \\ \dot{\bar{v}}_{11\,i} \\ \ddot{\bar{v}}_{10\,i} \\ \ddot{\bar{v}}_{11\,i} \end{bmatrix}}_{y_{1i}(\chi)} = \underbrace{\begin{bmatrix} -{h_0}^T- \\ -{h_1}^T- \\ -{\dot{h}_0}^T- \\ -{\dot{h}_1}^T- \\ -{\ddot{h}_0}^T- \\ -{\ddot{h}_1}^T- \end{bmatrix}}_{=:H} \cdot c_{1i} \tag{92}$$

or by computing it for all axes columnwise in matrix C_1 at the same time

$$\begin{bmatrix} {\bar{v}_{10}}^T \\ {\bar{v}_{11}}^T \\ {\dot{\bar{v}}_{10}}^T \\ {\dot{\bar{v}}_{11}}^T \\ {\ddot{\bar{v}}_{10}}^T \\ {\ddot{\bar{v}}_{11}}^T \end{bmatrix} = H \cdot \underbrace{\begin{bmatrix} c_{1x} & c_{1y} & c_{1z} \end{bmatrix}}_{=:C_1}, \text{where} \tag{93}$$

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the time matrix H has the dimension $6\,x\,(n+1)$. This means, to satisfy the boundary conditions exactly a polynomial of order 5 is needed.

Inserting Eqs. (30), (31), (44), (45), (76), (77) into Eq. (93) gives

$$\begin{bmatrix} \bar{v}_{10}{}^T \\ \bar{v}_{11}{}^T \\ \dot{\bar{v}}_{10p}{}^T + \chi_1 \cdot \bar{v}_{10}{}^T \\ \dot{\bar{v}}_{11p}{}^T + \chi_4 \cdot \bar{v}_{11}{}^T \\ \ddot{\bar{v}}_{10p}{}^T + [\chi_1 \quad \chi_7] \cdot \underbrace{\begin{bmatrix} u_{20\,1}{}^T \\ u_{20\,2}{}^T \end{bmatrix}}_{U_{20}} \\ \ddot{\bar{v}}_{11p}{}^T + [\chi_4 \quad \chi_{10}] \cdot \underbrace{\begin{bmatrix} u_{21\,1}{}^T \\ u_{21\,2}{}^T \end{bmatrix}}_{U_{21}} \end{bmatrix} = H \cdot \underbrace{[c_{1x} \quad c_{1y} \quad c_{1z}]}_{C_1}, \tag{94}$$

$$\underbrace{\begin{bmatrix} \bar{v}_{10}{}^T \\ \bar{v}_{11}{}^T \\ \dot{\bar{v}}_{10p}{}^T \\ \dot{\bar{v}}_{11p}{}^T \\ \ddot{\bar{v}}_{10p}{}^T \\ \ddot{\bar{v}}_{11p}{}^T \end{bmatrix}}_{=:V_0} + \left[\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{v}_{10x} & 0 & 0 & 0 \\ 0 & \bar{v}_{11x} & 0 & 0 \\ u_{20\,1x} & 0 & u_{20\,2x} & 0 \\ 0 & u_{21\,1x} & 0 & u_{21\,2x} \end{bmatrix}}_{=:U_x} \underbrace{\begin{bmatrix} \chi_1 \\ \chi_4 \\ \chi_7 \\ \chi_{10} \end{bmatrix}}_{=:\chi_v} \quad \begin{matrix} | \\ U_y \cdot \chi_v \\ | \end{matrix} \quad \begin{matrix} | \\ U_z \cdot \chi_v \\ | \end{matrix} \right] = H \cdot C_1. \tag{95}$$

The general solution of Eq. (95) is

$$C_1 = \underbrace{H^T(HH^T)^{-1}}_{=:H_1} \cdot V_0 + H^T(HH^T)^{-1} \cdot \begin{bmatrix} | & | & | \\ U_x \cdot \chi_v & U_y \cdot \chi_v & U_z \cdot \chi_v \\ | & | & | \end{bmatrix} + H_0 \cdot \begin{bmatrix} | & | & | \\ \rho_x & \rho_y & \rho_z \\ | & | & | \end{bmatrix}, \tag{96}$$

$$C_1 = H_1 \cdot \underbrace{[V_{0x} \quad V_{0y} \quad V_{0z}]}_{=V_0} + H_1 \cdot \begin{bmatrix} | & | & | \\ U_x \cdot \chi_v & U_y \cdot \chi_v & U_z \cdot \chi_v \\ | & | & | \end{bmatrix} + H_0 \cdot \begin{bmatrix} | & | & | \\ \rho_x & \rho_y & \rho_z \\ | & | & | \end{bmatrix}, \tag{97}$$

H_0 is the $(n+1)x(n-5)$ dimensional null space matrix of H, i.e. the matrix H_0 causes the matrix product

$$H \cdot H_0 = 0, \tag{98}$$

and thus all possible linear combinations of the columns of H_0 in each axis, i.e.

$$\underbrace{H \cdot H_0}_{0} \cdot \begin{bmatrix} | & | & | \\ \rho_x & \rho_y & \rho_z \\ | & | & | \end{bmatrix} = 0, \tag{99}$$

where ρ_x, ρ_y, ρ_z are the $(n-5)x1$ dimensional free null space component vectors in all axes. Eq. (97) can also be expressed in vector notation yielding

$$\begin{bmatrix} c_{1x} \\ c_{1y} \\ c_{1z} \end{bmatrix} = \begin{bmatrix} H_1 \cdot V_{0x} \\ H_1 \cdot V_{0y} \\ H_1 \cdot V_{0z} \end{bmatrix} + \begin{bmatrix} H_1 \cdot U_x & H_0 & 0 & 0 \\ H_1 \cdot U_y & 0 & H_0 & 0 \\ H_1 \cdot U_z & 0 & 0 & H_0 \end{bmatrix} \cdot \begin{bmatrix} \chi_v \\ \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}. \tag{100}$$

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Note that H_1 from Eqs. (96), (100) may be numerically difficult to compute. A measure to solve this numerical problem comes from Eqs. (20), (21), (22): Evaluating ${h_0}^T,\ {\dot{h}_0}^T,\ {\ddot{h}_0}^T$ for $t = 0$ in Eqs. (101), (103) allows to compute the last three coefficients corresponding to the smallest order in t, i.e. t^2, t^1, t^0:

$$\begin{bmatrix} {\ddot{\bar{v}}_{10p}}^T + [\chi_1 \quad \chi_7] \cdot U_{20} \\ {\dot{\bar{v}}_{10p}}^T + \chi_1 \cdot {\bar{v}_{10}}^T \\ {\bar{v}_{10}}^T \end{bmatrix} = \begin{bmatrix} -{\ddot{h}_0}^T - \\ -{\dot{h}_0}^T - \\ -{h_0}^T - \end{bmatrix} \cdot [c_{1x} \quad c_{1y} \quad c_{1z}] = \begin{bmatrix} 0 & \dots & 2 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} [c_{1x} \quad c_{1y} \quad c_{1z}] \\ = \begin{bmatrix} 2c_{1x2} & 2c_{1y2} & 2c_{1z2} \\ c_{1x1} & c_{1y1} & c_{1z1} \\ c_{1x0} & c_{1y0} & c_{1z0} \end{bmatrix}. \tag{101}$$

after rearranging terms a particular solution for the first three coefficients are summarized in matrix C_{1Qp}:

$$\underbrace{\begin{bmatrix} c_{1x2} & c_{1y2} & c_{1z2} \\ c_{1x1} & c_{1y1} & c_{1z1} \\ c_{1x0} & c_{1y0} & c_{1z0} \end{bmatrix}}_{=:C_{1Qp}} = \underbrace{\begin{bmatrix} \frac{1}{2}\ddot{\bar{v}}_{10p} \\ \dot{\bar{v}}_{10p} \\ \bar{v}_{10} \end{bmatrix}^T}_{=:C_{1Q0}} + \left[\underbrace{\begin{bmatrix} \frac{1}{2}u_{20\ 1x} & \frac{1}{2}u_{20\ 2x} \\ \bar{v}_{10x} & 0 \\ 0 & 0 \end{bmatrix}}_{U_{Qx}} \cdot \underbrace{\begin{bmatrix} \chi_1 \\ \chi_7 \end{bmatrix}}_{=:\chi_{vQ}} \quad \begin{matrix} | \\ U_{Qy} \cdot \chi_{vQ} \\ | \end{matrix} \quad \begin{matrix} | \\ U_{Qz} \cdot \chi_{vQ} \\ | \end{matrix} \right]. \tag{102}$$

Now, the rest of the coefficients, summarized in matrix C_{1Rp}, needs to be computed, which form a particular solution of the unknown coefficients in C_1, C_{1p}:

$$C_{1p} = \begin{bmatrix} C_{1Rp} \\ C_{1Qp} \end{bmatrix}. \tag{103}$$

C_{1Rp} corresponds to orders three and higher in t, i.e. $t^3, t^4, t^5 \dots t^n$, and can be determined as follows:
Eq. (103) is evaluated at the final boundary conditions only to yield

$$\begin{bmatrix} {\bar{v}_{11}}^T \\ {\dot{\bar{v}}_{11p}}^T + \chi_4 \cdot {\bar{v}_{11}}^T \\ {\ddot{\bar{v}}_{11p}}^T + [\chi_4 \quad \chi_{10}] \cdot \begin{bmatrix} {u_{21\ 1}}^T \\ {u_{21\ 2}}^T \end{bmatrix} \end{bmatrix} = \underbrace{\begin{bmatrix} -{h_1}^T - \\ -{\dot{h}_1}^T - \\ -{\ddot{h}_1}^T - \end{bmatrix}}_{=:H_R} \cdot \underbrace{\begin{bmatrix} C_{1Rp} \\ C_{1Qp} \end{bmatrix}} = \underbrace{[H_{Ra} \quad H_{Rb}]}_{=:H_R} \cdot \underbrace{\begin{bmatrix} C_{1Rp} \\ C_{1Qp} \end{bmatrix}}_{C_1}, \tag{104}$$

$$\underbrace{\begin{bmatrix} {\bar{v}_{11}}^T \\ {\dot{\bar{v}}_{11p}}^T \\ {\ddot{\bar{v}}_{11p}}^T \end{bmatrix}}_{V_{0R}} + \left[\underbrace{\begin{bmatrix} 0 & 0 \\ \bar{v}_{11x} & 0 \\ u_{21\ 1x} & u_{21\ 2x} \end{bmatrix}}_{U_{Rx}} \cdot \underbrace{\begin{bmatrix} \chi_4 \\ \chi_{10} \end{bmatrix}}_{=:\chi_{vR}} \quad \begin{matrix} | \\ U_{Ry} \cdot \chi_{vR} \\ | \end{matrix} \quad \begin{matrix} | \\ U_{Rz} \cdot \chi_{vR} \\ | \end{matrix} \right] - H_{Rb} \cdot C_{1Qp} = H_{Ra} \cdot C_{1Rp}. \tag{105}$$

Inserting Eq. (102) in Eq. (105) gives the matrix equation

$$V_{0R} + \begin{bmatrix} | & | & | \\ U_{Rx} \cdot \chi_{vR} & U_{Ry} \cdot \chi_{vR} & U_{Rz} \cdot \chi_{vR} \\ | & | & | \end{bmatrix} - H_{Rb} \begin{bmatrix} | & | & | \\ U_{Qx} \cdot \chi_{vQ} & U_{Qy} \cdot \chi_{vQ} & U_{Qz} \cdot \chi_{vQ} \\ | & | & | \end{bmatrix} - H_{Rb} C_{1Q0} = H_{Ra} \cdot C_{1Rp} \tag{106}$$

$$V_{0R} - H_{Rb} C_{1Q0} + \left[\begin{matrix} | \\ \underbrace{[U_{Rx} - H_{Rb} U_{Qx}]}_{=:U_{RQx}} \cdot \underbrace{\begin{bmatrix} \chi_{vR} \\ \chi_{vQ} \end{bmatrix}}_{=:\bar{\chi}_v} \\ | \end{matrix} \quad \begin{matrix} | \\ U_{RQy} \cdot \bar{\chi}_v \\ | \end{matrix} \quad \begin{matrix} | \\ U_{RQz} \cdot \bar{\chi}_v \\ | \end{matrix} \right] = H_{Ra} \cdot C_{1Rp} \tag{107}$$

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From Eq. (107) a particular solution C_{1Rp} can be computed in the usual manner:

$$C_{1Rp} = \underbrace{\underbrace{{H_{Ra}}^T\left(H_{Ra} \cdot {H_{Ra}}^T\right)^{-1}}_{=:H_{R1}}\left(V_{0R} - H_{Rb}C_{1Q0}\right)}_{=:C_{1R0}} + H_{R1} \cdot \begin{bmatrix} | & | & | \\ U_{RQx} \cdot \bar{\chi}_v & U_{RQy} \cdot \bar{\chi}_v & U_{RQz} \cdot \bar{\chi}_v \\ | & | & | \end{bmatrix} \tag{108}$$

Using the definition of $\bar{\chi}_v$ from Eq. (107) in Eq. (102) gives:

$$C_{1Qp} = C_{1Q0} + \begin{bmatrix} | & | & | \\ U_{Qx} \cdot \chi_{vQ} & U_{Qy} \cdot \chi_{vQ} & U_{Qz} \cdot \chi_{vQ} \\ | & | & | \end{bmatrix} = C_{1Q0} + \begin{bmatrix} & | & & | & & | \\ [0 & U_{Qx}] \cdot \bar{\chi}_v & [0 & U_{Qy}] \cdot \bar{\chi}_v & [0 & U_{Qz}] \cdot \bar{\chi}_v \\ & | & & | & & | \end{bmatrix} \tag{109}$$

Inserting Eqs. (117), (118) in Eq. (112) gives:

$$C_{1p} = \begin{bmatrix} C_{1Rp} \\ C_{1Qp} \end{bmatrix} = \underbrace{\begin{bmatrix} C_{1R0} \\ C_{1Q0} \end{bmatrix}}_{=:C_{10}} + \left[\underbrace{\begin{bmatrix} U_{RQx} \\ [0 \quad U_{Qx}] \end{bmatrix}}_{=:S_x} \cdot \bar{\chi}_v \quad \underbrace{\begin{bmatrix} U_{RQy} \\ [0 \quad U_{Qy}] \end{bmatrix}}_{=:S_y} \cdot \bar{\chi}_v \quad \underbrace{\begin{bmatrix} U_{RQz} \\ [0 \quad U_{Qz}] \end{bmatrix}}_{=:S_z} \cdot \bar{\chi}_v \right] \tag{110}$$

The general solution of describing all possible coefficients satisfying the boundary constraints in Eq. (94) is retained by adding the null space vector combinations to the particular solution in Eq. (110) as done in Eq. (96):

$$C_1 = C_{1p} + H_0 \cdot \begin{bmatrix} | & | & | \\ \rho_x & \rho_y & \rho_z \\ | & | & | \end{bmatrix} = C_{10} + \begin{bmatrix} S_x \cdot \bar{\chi}_v & S_y \cdot \bar{\chi}_v & S_z \cdot \bar{\chi}_v \end{bmatrix} + H_0 \cdot \begin{bmatrix} | & | & | \\ \rho_x & \rho_y & \rho_z \\ | & | & | \end{bmatrix}. \tag{111}$$

Using vector notation in Eq. (111), i.e.

$$c_1 = [c_{1x} \quad c_{1y} \quad c_{1z}] \qquad c_{10} = [c_{10x} \quad c_{10y} \quad c_{10z}], \tag{112}$$

results in

$$\underbrace{\begin{bmatrix} c_{1x} \\ c_{1y} \\ c_{1z} \end{bmatrix}}_{=:c_1} = \underbrace{\begin{bmatrix} c_{10x} \\ c_{10y} \\ c_{10z} \end{bmatrix}}_{=:c_{10}} + \underbrace{\begin{bmatrix} S_x & H_0 & 0 & 0 \\ S_y & 0 & H_0 & 0 \\ S_z & 0 & 0 & H_0 \end{bmatrix}}_{=:C_{v_1}} \cdot \underbrace{\begin{bmatrix} \bar{\chi}_v \\ \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}}_{=:x_{v_1}}, \tag{113}$$

$$c_1 = c_{10} + C_{v_1} \cdot x_{v_1}, \tag{114}$$

with the freely selectable $4 + 3(n - 5)$-dimensional parameter vector x_{v_1}.

Eq. (114) summarizes the result: The unknown coefficients of vector v_1 in Eq. (19) can be determined by a *linear* operation. Arbitrary numbers comprised in vector x_{v_1} are multiplied with a predefined matrix C_{v_1} and then added with a predifined vector c_{10} which depend on the boundary conditions.

Vector $\bar{v}_1(t)$ and its derivative defined in Eq. (21) can then be computed as

$$\bar{v}_1(t) = \begin{bmatrix} h(t)^T & 0 & 0 \\ 0 & h(t)^T & 0 \\ 0 & 0 & h(t)^T \end{bmatrix} \cdot c_1, \tag{115}$$

$$\dot{\bar{v}}_1(t) = \begin{bmatrix} \dot{h}(t)^T & 0 & 0 \\ 0 & \dot{h}(t)^T & 0 \\ 0 & 0 & \dot{h}(t)^T \end{bmatrix} \cdot c_1, \tag{116}$$

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Satisfaction of boundary conditions of contribution factors δ corresponding to vector v_2

For the remaining three polynomials $\delta_0(t) = h^T(t) \cdot c_j$ ($j = \alpha, \beta, \gamma$ from Eq. (24)) the vector boundary conditions according to Eqs. (34), (35), (56), (57), (84), (85) need to be satisfied by the unknown coefficients c_j. With vector algebraic manipulation the following general solution similar to Eq. (113) can be derived,

$$\underbrace{\begin{bmatrix} c_\alpha \\ c_\beta \\ c_\gamma \end{bmatrix}}_{=:c_\delta} = \underbrace{\begin{bmatrix} c_{\delta 0\alpha} \\ c_{\delta 0\beta} \\ c_{\delta 0\gamma} \end{bmatrix}}_{=:c_{\delta 0}} + \underbrace{\begin{bmatrix} S_{\delta\alpha} & H_0 & 0 & 0 \\ S_{\delta\beta} & 0 & H_0 & 0 \\ S_{\delta\gamma} & 0 & 0 & H_0 \end{bmatrix}}_{=:C_\delta} \cdot \underbrace{\begin{bmatrix} \bar{\chi}_\delta \\ \xi_\alpha \\ \xi_\beta \\ \xi_\gamma \end{bmatrix}}_{=:x_\delta}, \tag{117}$$

$$c_\delta = c_{\delta 0} + C_\delta \cdot x_\delta, \tag{118}$$

with the freely selectable $10 + 3(n-5)$-dimensional parameter vector x_δ, in which $\bar{\chi}_\delta$

$$\bar{\chi}_\delta = \begin{bmatrix} \chi_{\delta R} \\ \chi_{\delta Q} \end{bmatrix}, \tag{119}$$

where $\chi_{\delta R}$, $\chi_{\delta Q}$ are defined in Eqs. (84),(85), vectors ξ_α, ξ_β, ξ_γ are for each axis the free selectable components of the null space columns vectors of matrix H_0, vector $c_{\delta 0}$ and matrix C_δ are similar to Eqs. (114) depending on the initial and final conditions.

Eq. (118) summarizes the result: The unknown coefficients of vector v_2 in Eq. (23) consisting of three vectors perpendicular to v_1 weighted with polynomials with coefficients denoted c_δ which can be determined by a *linear* operation. Arbitrary numbers comprised in vector x_δ are multiplied with a predefined matrix C_δ and then added with a predifined vector $c_{\delta 0}$ which depends on the boundary conditions.

Vector $\bar{v}_2(t)$ and its derivative defined in Eq. (23) can then be computed as

$$\bar{v}_2 = \widetilde{\bar{v}_1} \cdot \delta = \widetilde{\bar{v}_1} \cdot \begin{bmatrix} h(t)^T & 0 & 0 \\ 0 & h(t)^T & 0 \\ 0 & 0 & h(t)^T \end{bmatrix} \cdot c_\delta. \tag{120}$$

$$\dot{\bar{v}}_2 = \dot{\widetilde{\bar{v}_1}} \cdot \delta + \widetilde{\bar{v}_1} \cdot \dot{\delta} = \dot{\widetilde{\bar{v}_1}} \cdot \begin{bmatrix} h(t)^T & 0 & 0 \\ 0 & h(t)^T & 0 \\ 0 & 0 & h(t)^T \end{bmatrix} \cdot c_\delta + \widetilde{\bar{v}_1} \cdot \begin{bmatrix} \dot{h}(t)^T & 0 & 0 \\ 0 & \dot{h}(t)^T & 0 \\ 0 & 0 & \dot{h}(t)^T \end{bmatrix} \cdot c_\delta. \tag{121}$$

$$\dot{\bar{v}}_2 = \left[\dot{\widetilde{\bar{v}_1}} \cdot \begin{bmatrix} h(t)^T & 0 & 0 \\ 0 & h(t)^T & 0 \\ 0 & 0 & h(t)^T \end{bmatrix} + \widetilde{\bar{v}_1} \cdot \begin{bmatrix} \dot{h}(t)^T & 0 & 0 \\ 0 & \dot{h}(t)^T & 0 \\ 0 & 0 & \dot{h}(t)^T \end{bmatrix} \right] \cdot c_\delta. \tag{122}$$

E. Computation of key properties: rate, angular momentum and torque

Once the coefficients in Eqs. (114), (118) have been selected – for instance by methods in chapter IV - the rate signal ω can be derived from the orthonormal vectors v_1, v_2, v_3 as shown in Eq. (17):

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{v_3}^T \cdot v_2 & = & -\dot{v_2}^T \cdot v_3 \\ \dot{v_1}^T \cdot v_3 & = & -\dot{v_3}^T \cdot v_1 \\ \dot{v_2}^T \cdot v_1 & = & -\dot{v_1}^T \cdot v_2 \end{bmatrix}. \tag{123}$$

From the rate signal ω the angular momentum h^b can be expressed in body frame as

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$$h^b = I \cdot T_i^b \cdot \omega^i, \tag{124}$$

where I denotes the 3x3 moment of inertia matrix. Inserting Eqs. (11), (123) into Eq. (124) yields for the angular momentum

$$h^b = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} -{v_3}^T \cdot \dot{v}_2 \\ +{v_3}^T \cdot \dot{v}_1 \\ -{v_2}^T \cdot \dot{v}_1 \end{bmatrix}. \tag{125}$$

From the angular momentum h^b the torque signal τ can be derived by building the time derivative (denoted as "$(\ )'$"with respect to the *inertial* frame as shown in Fig. 1 which gives

$$\tau^b = (h^b)' = I \cdot \left(T_i^b \cdot \omega^i\right)' = I \cdot \left(\left(T_i^b\right)' \cdot \omega^i + T_i^b \cdot (\omega^i)'\right). \tag{126}$$

From Eq. (4) follows

$$\left(T_b^i\right)' = \widetilde{\omega}^i \cdot T_b^i \rightarrow \left(\left(T_b^i\right)'\right)^T = \left(T_i^b\right)' = \left(\widetilde{\omega}^i \cdot T_b^i\right)^T = -T_b^i \cdot \widetilde{\omega}^i \rightarrow \left(T_i^b\right)' = -T_b^i \cdot \widetilde{\omega}^i. \tag{127}$$

Inserting Eq. (127) into Eq. (126) gives

$$\tau^b = I \cdot \left(\left(T_i^b\right)' \cdot \omega^i + T_i^b \cdot (\omega^i)'\right) = I \cdot \left(\underbrace{-T_b^i \cdot \widetilde{\omega}^i \cdot \omega^i}_{=0} + T_i^b \cdot (\omega^i)'\right) = I \cdot T_i^b \cdot (\omega^i)'. \tag{128}$$

In Eq. (128) the rate acceleration $(\omega^i)'$ can be computed from Eq. (18) to get

$$\tau^b = I \cdot T_i^b \cdot (\omega^i)' = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} +{\ddot{v}_3}^T \cdot v_2 + {\dot{v}_3}^T \cdot \dot{v}_2 \\ +{\ddot{v}_1}^T \cdot v_3 + {\dot{v}_1}^T \cdot \dot{v}_3 \\ -{\ddot{v}_1}^T \cdot v_2 - {\dot{v}_1}^T \cdot \dot{v}_2 \end{bmatrix}. \tag{129}$$

F. Numerical computation of the time dependent unit vectors v_1, v_2, v_3 and its derivatives

In order to compute the key properties rate, angular momentum and torque from Eqs. (123), (125), (129), the computation of the time dependent unit vectors $v_1(t)$, $v_2(t)$, $v_3(t)$, its first derivatives $\dot{v}_1(t)$, $\dot{v}_2(t)$, $\dot{v}_3(t)$ and its second derivatives $\ddot{v}_1(t)$, $\ddot{v}_2(t)$, $\ddot{v}_3(t)$ is required.

From Eq. (19) vector v_1 is defined as

$$v_1 = \frac{\bar{v}_1}{|\bar{v}_1|} = \frac{\bar{v}_1}{\sqrt{\underbrace{{\bar{v}_1}^T \bar{v}_1}_{=:A}}} = \frac{\bar{v}_1}{\sqrt{A}}. \tag{130}$$

The derivative can be computed from the quotient rule to be

$$\dot{v}_1 = \frac{\dot{\bar{v}}_1 \sqrt{A} - \bar{v}_1 \dot{\sqrt{A}}}{A}. \tag{131}$$

Using Eq. (176) yields for $\dot{\sqrt{A}}$

$$\dot{\sqrt{A}} = \frac{1}{2\sqrt{A}} \cdot \dot{A} = \frac{2{\dot{\bar{v}}_1}^T \bar{v}_1}{2\sqrt{{\bar{v}_1}^T \bar{v}_1}} = \frac{{\dot{\bar{v}}_1}^T \bar{v}_1}{\sqrt{{\bar{v}_1}^T \bar{v}_1}} = \frac{{\dot{\bar{v}}_1}^T \bar{v}_1}{\sqrt{A}}, \tag{132}$$

and inserting Eq. (132) into Eq. (131) gives for $\dot{v}_1$

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$$\dot{v}_1 = \frac{\dot{\bar{v}}_1\sqrt{A} - \bar{v}_1\dot{\sqrt{A}}}{A} = \frac{\dot{\bar{v}}_1\sqrt{A} - \bar{v}_1\dfrac{\dot{\bar{v}}_1{}^T\bar{v}_1}{\sqrt{A}}}{A} = \frac{\dot{\bar{v}}_1\bar{v}_1{}^T\bar{v}_1 - \bar{v}_1\dot{\bar{v}}_1{}^T\bar{v}_1}{A^{\frac{3}{2}}} = \frac{\overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}^{=:E}\cdot\bar{v}_1{}^T\bar{v}_1 - \bar{v}_1\bar{v}_1{}^T}{A^{\frac{3}{2}}}\dot{\bar{v}}_1, \tag{133}$$

$$\dot{v}_1 = \frac{E\cdot\bar{v}_1{}^T\bar{v}_1 - \bar{v}_1\bar{v}_1{}^T}{A^{\frac{3}{2}}}\dot{\bar{v}}_1. \tag{134}$$

Vector $\dot{\bar{v}}_1$ is computed from Eq. (20). The second derivative, $\ddot{v}_1$, is derived from Eq. (133):

$$\ddot{v}_1 = \frac{\left(\ddot{\bar{v}}_1\sqrt{A} + \dot{\bar{v}}_1\dot{\sqrt{A}} - \dot{\bar{v}}_1\dot{\sqrt{A}} - \bar{v}_1\ddot{\sqrt{A}}\right)A - \left(\dot{\bar{v}}_1\sqrt{A} - \bar{v}_1\dot{\sqrt{A}}\right)\dot{A}}{A^2}, \tag{135}$$

$$\ddot{v}_1 = \frac{\left(\ddot{\bar{v}}_1\sqrt{A} - \bar{v}_1\ddot{\sqrt{A}}\right)A - \left(\dot{\bar{v}}_1\sqrt{A} - \bar{v}_1\dot{\sqrt{A}}\right)\dot{A}}{A^2}, \tag{136}$$

$$\ddot{v}_1 = \frac{\ddot{\bar{v}}_1 A^{\frac{3}{2}} - \dot{\bar{v}}_1\sqrt{A}\cdot\dot{A} - \bar{v}_1\left(\ddot{\sqrt{A}}\cdot A - \dot{\sqrt{A}}\cdot\dot{A}\right)}{A^2}. \tag{137}$$

and vector $\ddot{\bar{v}}_1$ is computed from Eq. (22). In Eq. (137) the expression $\ddot{\sqrt{A}}$ can be computed from Eq. (132)

$$\ddot{\sqrt{A}} = \frac{\left(\ddot{\bar{v}}_1{}^T\bar{v}_1 + \dot{\bar{v}}_1{}^T\dot{\bar{v}}_1\right)\sqrt{A} - \dot{\bar{v}}_1{}^T\bar{v}_1\dot{\sqrt{A}}}{A}, \tag{138}$$

and using Eq. (132)

$$\ddot{\sqrt{A}} = \frac{\left(\ddot{\bar{v}}_1{}^T\bar{v}_1 + \dot{\bar{v}}_1{}^T\dot{\bar{v}}_1\right)\sqrt{A} - \dot{\bar{v}}_1{}^T\bar{v}_1\dfrac{\dot{\bar{v}}_1{}^T\bar{v}_1}{\sqrt{A}}}{A}, \tag{139}$$

$$\ddot{\sqrt{A}} = \frac{\left(\ddot{\bar{v}}_1{}^T\bar{v}_1 + \dot{\bar{v}}_1{}^T\dot{\bar{v}}_1\right)A - \dot{\bar{v}}_1{}^T\bar{v}_1\dot{\bar{v}}_1{}^T\bar{v}_1}{A^{\frac{3}{2}}} = \frac{\left(\ddot{\bar{v}}_1{}^T\bar{v}_1 + \dot{\bar{v}}_1{}^T\dot{\bar{v}}_1\right)A - \left(\dot{\bar{v}}_1{}^T\bar{v}_1\right)^2}{A^{\frac{3}{2}}}. \tag{140}$$

Vectors v_1, $\dot{v}_1$, $\ddot{v}_1$ are fully determined now.

The computation of these vectors vectors v_2, $\dot{v}_2$, $\ddot{v}_2$ is similar to the computation of v_1, $\dot{v}_1$, $\ddot{v}_1$ and is achieved by substituting in the contained equations index 1 with index 2. The only difference is that vectors $\bar{v}_2$, $\dot{\bar{v}}_2$ and $\ddot{\bar{v}}_2$ are computed by Eqs. (50)-(52) instead.

A similar computation of vectors v_3, $\dot{v}_3$, $\ddot{v}_3$ causes numerical problems. The computation of those vectors using vectors v_1, $\dot{v}_1$, $\ddot{v}_1$, v_2, $\dot{v}_2$, $\ddot{v}_2$ in Eqs. (50)-(52) is numerically better.

IV. Optimization with Least-Squares-Approach

One possible strategy in the design of slew maneuvers is to have a flow of an angular momentum which changes as low as possible over a big time span from its predefines start value towards its destination value. A least squares criterion on the angular momentum components naturally balances the magnitude over the slew time and leads to small time instances of high changes at the begin and the end of the slew which will then appear as high torques.

The angular momentum h^b is computed according to Eq. (125), (129) as

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$$h^b = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} -{v_3}^T \cdot \dot{v}_2 \\ +{v_3}^T \cdot \dot{v}_1 \\ -{v_2}^T \cdot \dot{v}_1 \end{bmatrix}. \tag{141}$$

and the torque signal according to Eq. (129)

$$\tau^b = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} +{\ddot{v}_3}^T \cdot v_2 + {\dot{v}_3}^T \cdot \dot{v}_2 \\ +{\ddot{v}_1}^T \cdot v_3 + {\dot{v}_1}^T \cdot \dot{v}_3 \\ -{\ddot{v}_1}^T \cdot v_2 - {\dot{v}_1}^T \cdot \dot{v}_2 \end{bmatrix}. \tag{142}$$

Clearly, the products of type $v_1 \cdot {v_3}^T \cdot \dot{v}_2$ containing the unknown parameters enter highly *nonlinear* in the computation of angular momentum h^b as well as for the torque signal τ^b. Therefore, a direct manipulation of the components of h^b cannot be performed with the Least-Squares-approach.

However, inserting Eq. (133) to substitute $\dot{v}_1$ and similar $\dot{v}_2$ in Eq. (141) yields

$$h^b = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} -{v_3}^T \cdot \left(\dfrac{\dot{\bar{v}}_2 |\bar{v}_2| - \bar{v}_2 |\dot{\bar{v}}_2|}{|\bar{v}_2|^2} \right) \\ +{v_3}^T \cdot \left(\dfrac{\dot{\bar{v}}_1 |\bar{v}_1| - \bar{v}_1 |\dot{\bar{v}}_1|}{|\bar{v}_1|^2} \right) \\ -{v_2}^T \cdot \left(\dfrac{\dot{\bar{v}}_1 |\bar{v}_1| - \bar{v}_1 |\dot{\bar{v}}_1|}{|\bar{v}_1|^2} \right) \end{bmatrix} = I \cdot \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} -{v_3}^T \cdot \left(\dfrac{\dot{\bar{v}}_2}{|\bar{v}_2|} \right) \\ +{v_3}^T \cdot \left(\dfrac{\dot{\bar{v}}_1}{|\bar{v}_1|} \right) \\ -{v_2}^T \cdot \left(\dfrac{\dot{\bar{v}}_1}{|\bar{v}_1|} \right) \end{bmatrix}. \tag{143}$$

This means, that nevertheless the length of the angular momentum vector h^b can be reduced by minimizing $|\dot{\bar{v}}_1|$ and $|\dot{\bar{v}}_2|$ separately.

Inserting Eq. (114) in Eq. (116) results in

$$\dot{\bar{v}}_1(t) = \begin{bmatrix} \dot{h}(t)^T & 0 & 0 \\ 0 & \dot{h}(t)^T & 0 \\ 0 & 0 & \dot{h}(t)^T \end{bmatrix} \cdot \left(c_{10} + C_{v_1} \cdot x_{v_1} \right). \tag{144}$$

Evaluating Eq. (144) for several time instances $t_1, t_2, \dots t_m$ yields

$$\begin{aligned} \dot{\bar{v}}_1(t_1) &= \begin{bmatrix} \dot{h}(t_1)^T & 0 & 0 \\ 0 & \dot{h}(t_1)^T & 0 \\ 0 & 0 & \dot{h}(t_1)^T \end{bmatrix} \cdot \left(c_{10} + C_{v_1} \cdot x_{v_1} \right) \\ \dot{\bar{v}}_1(t_2) &= \begin{bmatrix} \dot{h}(t_2)^T & 0 & 0 \\ 0 & \dot{h}(t_2)^T & 0 \\ 0 & 0 & \dot{h}(t_2)^T \end{bmatrix} \cdot \left(c_{10} + C_{v_1} \cdot x_{v_1} \right) \\ \vdots \quad & \quad \vdots \qquad\qquad\qquad \vdots \\ \dot{\bar{v}}_1(t_m) &= \begin{bmatrix} \dot{h}(t_m)^T & 0 & 0 \\ 0 & \dot{h}(t_m)^T & 0 \\ 0 & 0 & \dot{h}(t_m)^T \end{bmatrix} \cdot \left(c_{10} + C_{v_1} \cdot x_{v_1} \right) \end{aligned} . \tag{145}$$

Wishing that the left hand side of Eq. (145) becomes zero introduces an equation error vector e:

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$$\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix} \dot{h}(t_1)^T & 0 & 0 \\ 0 & \dot{h}(t_1)^T & 0 \\ 0 & 0 & \dot{h}(t_1)^T \\ \dot{h}(t_2)^T & 0 & 0 \\ 0 & \dot{h}(t_2)^T & 0 \\ 0 & 0 & \dot{h}(t_2)^T \\ & \vdots & \\ \dot{h}(t_m)^T & 0 & 0 \\ 0 & \dot{h}(t_m)^T & 0 \\ 0 & 0 & \dot{h}(t_m)^T \end{bmatrix} \left(c_{10} + C_{v_1} \cdot x_{v_1}\right) + e. \tag{146}$$

and after rearranging terms

$$\underbrace{-\begin{bmatrix} \dot{h}(t_1)^T & 0 & 0 \\ 0 & \dot{h}(t_1)^T & 0 \\ 0 & 0 & \dot{h}(t_1)^T \\ \dot{h}(t_2)^T & 0 & 0 \\ 0 & \dot{h}(t_2)^T & 0 \\ 0 & 0 & \dot{h}(t_2)^T \\ & \vdots & \\ \dot{h}(t_m)^T & 0 & 0 \\ 0 & \dot{h}(t_m)^T & 0 \\ 0 & 0 & \dot{h}(t_m)^T \end{bmatrix} c_{10}}_{=:\tilde{y}_1} = \underbrace{\begin{bmatrix} \dot{h}(t_1)^T & 0 & 0 \\ 0 & \dot{h}(t_1)^T & 0 \\ 0 & 0 & \dot{h}(t_1)^T \\ \dot{h}(t_2)^T & 0 & 0 \\ 0 & \dot{h}(t_2)^T & 0 \\ 0 & 0 & \dot{h}(t_2)^T \\ & \vdots & \\ \dot{h}(t_m)^T & 0 & 0 \\ 0 & \dot{h}(t_m)^T & 0 \\ 0 & 0 & \dot{h}(t_m)^T \end{bmatrix}}_{=:H_1} C_{v_1} \cdot x_{v_1} + e. \tag{147}$$

in which clearly the structure evolves for which the Least-Squares approach [5] can be applied to find the optimal coefficients $\hat{x}_{v_1}$ by the well known solution,

$$\hat{x}_{v_1} = ({H_1}^T \cdot H_1)^{-1} \cdot {H_1}^T \cdot \tilde{y}_1. \tag{148}$$

With these coefficients the dynamics of vector $\bar{v}_1(t)$ in Eq. (124) and thus vector $v_1(t)$ in Eq. (19) and its derivatives are fixed. Now vector $v_2(t)$ needs to be determined:

In a similar way optimal values for c_δ can be found using Eqs. (19) to find the optimal coefficients $\hat{x}_\delta$:

$$\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\tilde{\bar{v}}}_1(t_1) \cdot \begin{bmatrix} h(t_1)^T & 0 & 0 \\ 0 & h(t_1)^T & 0 \\ 0 & 0 & h(t_1)^T \end{bmatrix} + \tilde{\bar{v}}_1(t_1) \cdot \begin{bmatrix} \dot{h}(t_1)^T & 0 & 0 \\ 0 & \dot{h}(t_1)^T & 0 \\ 0 & 0 & \dot{h}(t_1)^T \end{bmatrix} \\ \dot{\tilde{\bar{v}}}_1(t_2) \cdot \begin{bmatrix} h(t_2)^T & 0 & 0 \\ 0 & h(t_2)^T & 0 \\ 0 & 0 & h(t_2)^T \end{bmatrix} + \tilde{\bar{v}}_1(t_2) \cdot \begin{bmatrix} \dot{h}(t_2)^T & 0 & 0 \\ 0 & \dot{h}(t_2)^T & 0 \\ 0 & 0 & \dot{h}(t_2)^T \end{bmatrix} \\ \vdots \\ \dot{\tilde{\bar{v}}}_1(t_m) \cdot \begin{bmatrix} h(t_m)^T & 0 & 0 \\ 0 & h(t_m)^T & 0 \\ 0 & 0 & h(t_m)^T \end{bmatrix} + \tilde{\bar{v}}_1(t_m) \cdot \begin{bmatrix} \dot{h}(t_m)^T & 0 & 0 \\ 0 & \dot{h}(t_m)^T & 0 \\ 0 & 0 & \dot{h}(t_m)^T \end{bmatrix} \end{bmatrix}}_{=:J} (c_{\delta 0} + C_\delta \cdot x_\delta) + e, \tag{149}$$

and after rearranging terms

$$\underbrace{-J \cdot c_{\delta 0}}_{=:\tilde{y}_\delta} = \underbrace{J \cdot C_\delta}_{=:H_\delta} \cdot x_\delta + e. \tag{150}$$

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in which also the structure evolves for which the Least-Squares-Approach [5] can be applied to find the remaining optimal coefficients $\hat{x}_\delta$,

$$\hat{x}_\delta = ({H_\delta}^T \cdot H_\delta)^{-1} \cdot {H_\delta}^T \cdot \tilde{y}_\delta. \tag{151}$$

Numerical tests show that the matrices H_1 and H_δ have full rank, meaning that *all* free parameters in x_{v_1} and x_δ contribute to an optimal result.

With these coefficients the dynamics of vector $\bar{v}_2(t)$ in Eq. (121) and thus vector $v_2(t)$ in Eq. (23) and its derivatives are fixed. The dynamics of the attitude is now completely determined. The key signals like rate, angular momentum and torque can be computed from these vectors according to section D.

To summarize, In order to perform the Least-Squares-Approach the following computational steps need to be performed as described in Table 1:

Table 1: Computational steps to perform the Least-Squares-Approach

Step #	*Contents*
Step 1:	Polynomial order selection: Select order n of the polynomial, i.e. the dimension $n+1$ of time vector h in Eq. (20)
Step 2:	Mapping of physical boundary constraints into inertial frame i as vector boundary constraints $\bar{v}_1(t)$: Computation of vector c_{10} and matrix C_{v_1} in Eq. (114)
Step 3:	Solving Least-Squares problem as in Eq. (148)
Step 4:	Computation of vectors v_1, $\dot{v}_1$, $\ddot{v}_1$: The computations are performed in section 11.
Step 5:	Mapping of physical boundary constraints into inertial frame i as vector boundary constraints $\delta(t)$: Computation of vector $c_{\delta 0}$ and matrix C_δ in Eq. (120)
Step 6:	Solving Least-Squares problem as in Eq. (151)
Step 7:	Computation of vectors v_2, $\dot{v}_2$, $\ddot{v}_2$ as well as v_3, $\dot{v}_3$, $\ddot{v}_3$: The computations are performed in section G.
Step 8:	Computation of physical key signals: Time dependent computation of transformation matrix $T_b^i(t)$ in section 4, rate $\omega(t)$, angular momentum $h^b(t)$ and torque $\tau^b(t)$ in section F.

V.Numerical Example

Find in Table 2 the boundary conditions of a test scenario for which the presented method has been applied. The results show that the attitude boundary condition, the rate velocity and rate accelerations are satisfied in Fig. 4, Fig. 5, Fig. 6, respectively. The angular momentum has been minimized (Fig. 7) and the necessary torque in Fig. 3 shows for a very short time (below 1 second) a high peak value that can be realized by CMGs.

Table 2: Boundary conditions of slew test scenario

		Initial condition	**Final condition**
Attitude (Euler angle φ, θ, ψ in sequence 1-2-3) / rad	φ	-1.02184733442150	-0.87720143423133
	θ	0.92268904597360	-0.00006732457341
	ψ	-0.00626370681618	-0.00095302845659
Rotational rate ω / rad / s	ω_x	0.00000386474988	0.00001556724309
	ω_y	-0.00000173819461	-0.98920919739378
	ω_z	-0.81762488957959	0.91210161161081
Rate acceleration $\dot{\omega}$ / rad / s^2	$\dot{\omega}_x$	-0.00006614697502	-0.00661803211876
	$\dot{\omega}_y$	-0.00094028867769	-0.00000350937716
	$\dot{\omega}_z$	0.00002671368988	-0.00000010072336

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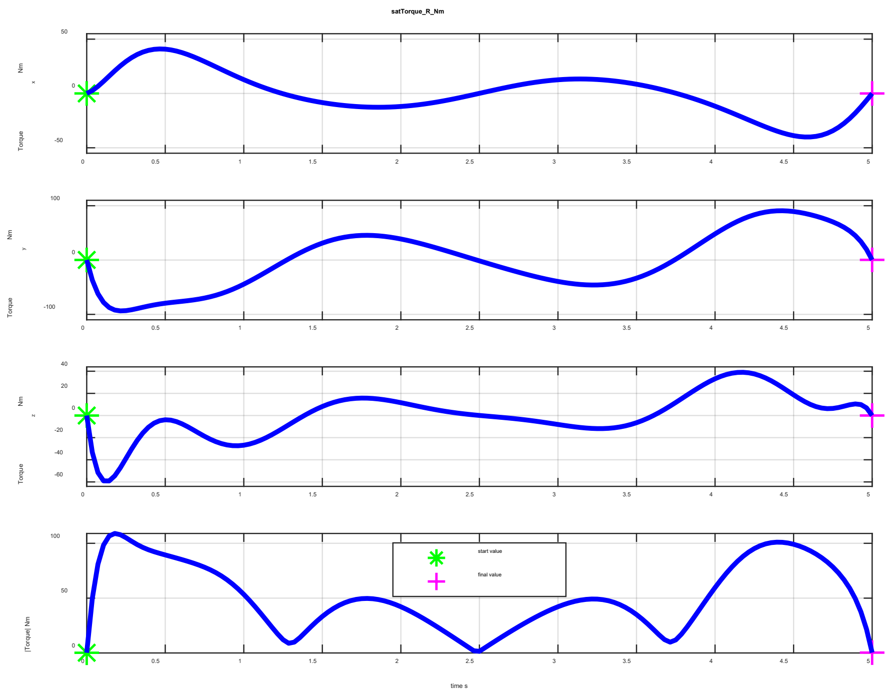

Fig. 3 Torque in body frame

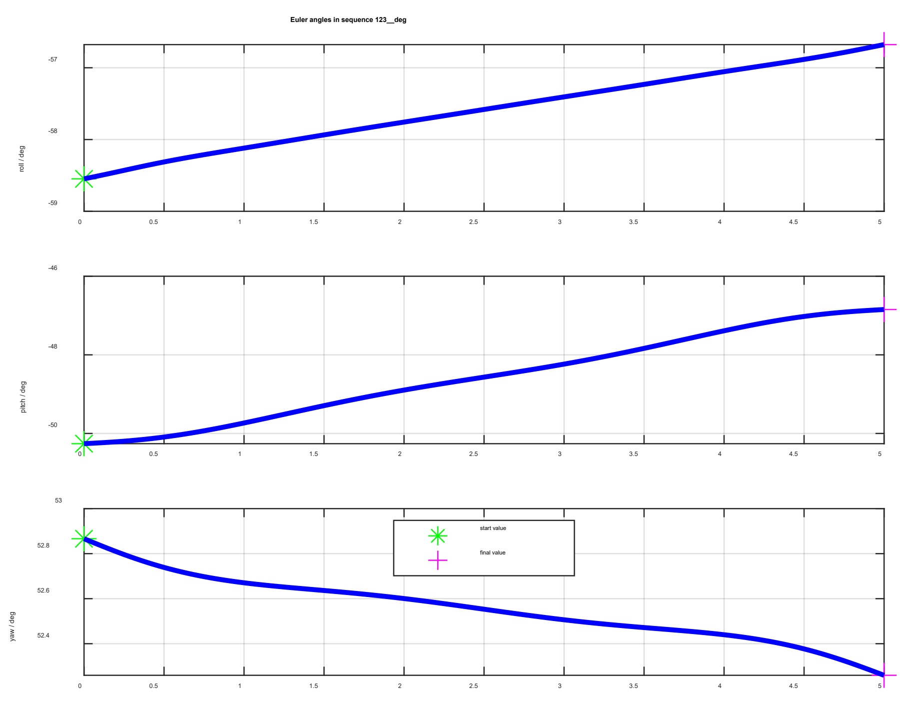

Fig. 4 Euler angles in sequence 123

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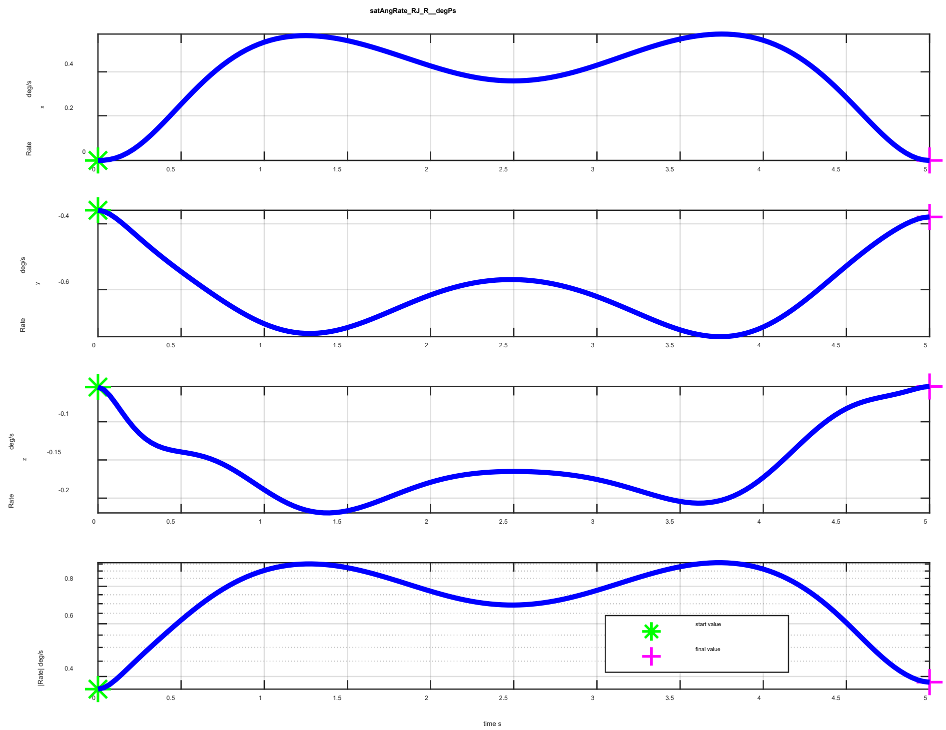

Fig. 5 Rotation rate velocity in body frame

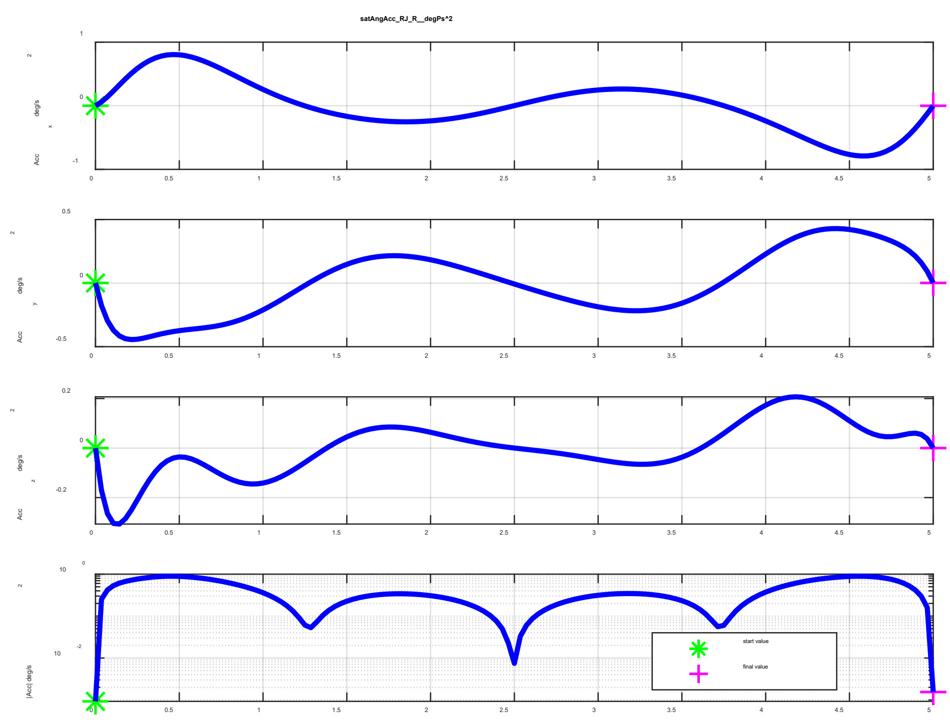

Fig. 6 Rotation rate acceleration in body frame

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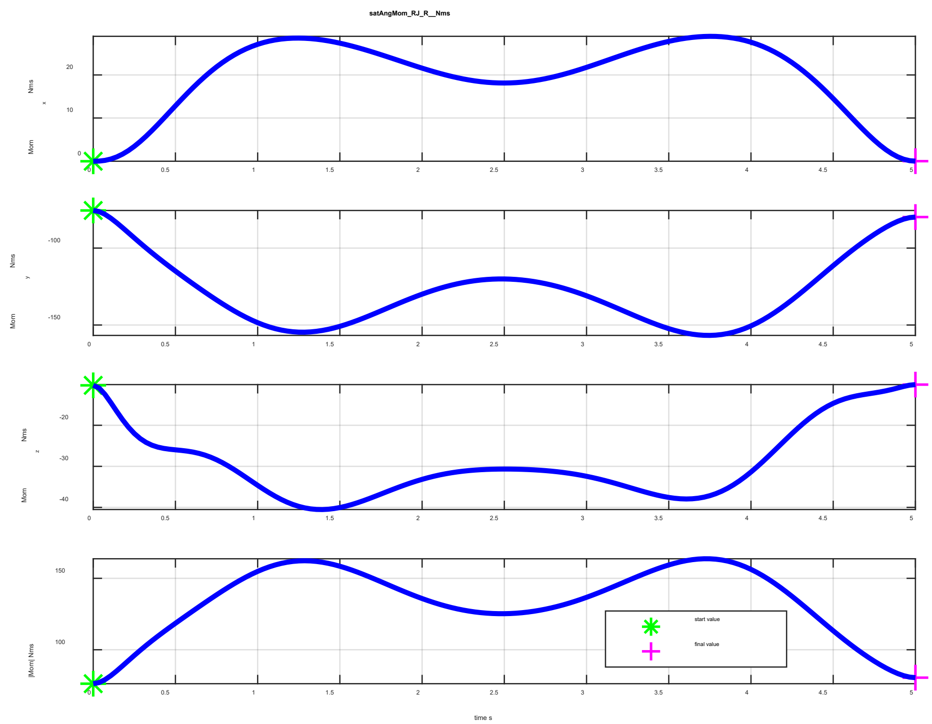

Fig. 7 Angular momentum in body frame

The overall computation time of the method is below 0.1 s using a standard laptop with an Intel(R) Core(TM) i7-8850H CPU @ 2.60GHz.

VI.Summary

In section I a parametrization of possible slew maneuvers has been presented which has the following properties:

1. The kinematic differential equation (4) is automatically *exactly* satisfied without the need of time consuming numerical integration.
2. In addition, the initial and the final boundary conditions in section 2 on attitude, rotational rate and rotational acceleration are automatically satisfied as well.
3. Fully free design parameters allow the shape optimization of the slew dynamics in between the boundary conditions.

The key idea of this parametrization is to map the *physical* boundary conditions from section 2 into *vector* boundary conditions for vector v_1 and for vector v_2 in section C. This mapping has two advantages: The *vector* boundary conditions can be satisfied *subsequently* rather than simultaneously, and additional unconstraint design parameters appear which can be used to optimize the slew shape. Of course, the proposed parametrization includes rest-to-rest maneuvers as well, if the rate velocity can be selected to be zero and the rate acceleration constraint is omitted.

On the basis of this parametrization a Least-Squares-Approach (section H) minimizes certain linear terms which appear in the formula for the angular momentum in Eq. (125). The *advantage* of the method is its very low computation time, since only a low number of analytical matrix/vector computations and no iterations need to be performed. Furthermore, only short time instances with high torques are generated. The *disadvantage* is that constraints for instance on the torque signal cannot be considered.

The test of the method applied in section V on a slew with given boundary conditions was successful in terms of satisfying all those boundary conditions, a small period of high torque and a low computation time.

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